CHALLENGE PROBLEM SET: CHAPTER 4, SECTION 1, COURSE WEEK 7

110.201 LINEAR ALGEBRA
PROFESSOR RICHARD BROWN

Question 1. In each instance (or working in three subgroups), determine which of the following subsets V of $\mathbb{R}^{3\times 3}$ are linear subspaces of $\mathbb{R}^{3\times 3}$:

(a) The diagonal 3×3 matrices,
(b) The upper triangular 3×3 matrices,
(c) The 3×3 matrices whose entries are all greater or equal to zero.

Question 2. Find a basis for each of the subspaces defined below, and determine the dimensions:

(a) The space of al polynomials $f(t) \in P_2$ such that $f(1) = 0$,
(b) The space of all polynomials $f(t) \in P_2$ such that $f(1) = 0$ and $f'(1) = 0$,
(c) The space of all polynomials $f(t) \in P_2$ such that $f(1) = 0$ and $\displaystyle\int_0^1 f(t)\, dt = 0$.

Question 3. In each instance (or working in three subgroups), find a basis for the subspace defined below and determine its dimension:

(a) The space of all 2×2 matrices $\mathbf{A}$ that comute with $\mathbf{B} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$,
(b) The space of all 2×2 matrices $\mathbf{A}$ such that $\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \mathbf{A} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$,
(c) The space of all 2×2 matrices $\mathbf{S}$ such that $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \mathbf{S} = \mathbf{S} \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$.

Question 4. If $\mathbf{c}$ is any vector in $\mathbb{R}^n$, what are the possible dimensions of the space V of all $n \times n$ matrices $\mathbf{A}$ such that $\mathbf{A}\mathbf{c} = \mathbf{0}$.

Question 5. A function $f(t) \in F(\mathbb{R}, \mathbb{R})$ is called even if $f(-t) = f(t)$, for all $t \in \mathbb{R}$, and odd if $f(-t) = -f(t)$, for all $t \in \mathbb{R}$. Do the following:

(a) Separately determine if the even functions and odd functions are each subspaces of $F(\mathbb{R}, \mathbb{R})$. Justify your answers carefully.

(b) Find a basis for each of the following linear spaces, and determine their dimensions:

(i) $\left\{ f \in P_4 \;\middle|\; f \text{ is even} \right\}$, (ii) $\left\{ f \in P_4 \;\middle|\; f \text{ is even} \right\}$.

Question 6. Find a basis of the space V of all 3×3 matrices $\mathbf{A}$ that commute with

$$B = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Question 7. Show that in an n-dimensional linear space we can find at most n linearly independent elements. *Hint:* Consider the proof of Theorem 4.1.5.

Question 8. Questions to argue over:

(a) The space $\mathbb{R}^{2\times 3}$ is 5 dimensional.

(b) All bases of P_3 contain at least one polynomial of degree ≤ 2.

(c) Every polynomial of degree 3 can be expressed as a linear combination of the polynomials $(t-3)$, $(t-3)^2$, and $(t-3)^3$.

(d) If a linear space can be spanned by 10 elements, then the dimension of V must be ≤ 10.

(e) There exists a basis of $\mathbb{R}^{2\times 2}$ consisting of four invertible matrices.

(f) If W is a subspace of V, and W is finite dimensional, then V must be finite dimensional as well.

(g) Any four-dimensional linear space has infinitely many three three-dimensional subspaces.