CHALLENGE PROBLEM SET: CHAPTER 3, SECTION 1-3, COURSE WEEK 5

110.201 LINEAR ALGEBRA
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Question 1. Describe the image and the kernel of the following matrices as subspaces of $\mathbb{R}^3$ by (1) identifying any redundant vectors among the columns of $\mathbf{A}$, (2) writing out all nontrivial relations among them, and (3) using this to locate vectors in the kernel. Then write a basis for each and note whether they are a point, line, plane or all of $\mathbb{R}^3$:

$$\text{(a)} \quad \mathbf{A} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 5 \end{bmatrix}, \quad \text{(b)} \quad \mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \\ 3 & 2 & 1 \end{bmatrix}, \quad \text{(c)} \quad \mathbf{A} = \begin{bmatrix} 1 & 3 & 6 \\ 1 & 2 & 5 \\ 1 & 1 & 4 \end{bmatrix}.$$

Question 2. Divide your group into three groups, where each group works concurrently on the board on one of the three following problems. In turn, present your solutions and contrast your solution with the others in this group. Discuss.

(a) Construct an example of a matrix $\mathbf{A}$ such that the image of $\mathbf{A}$ is the plane with normal vector $\begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} \in \mathbb{R}^3$.

(b) Construct an example of a linear transformation whose image is the line spanned by the vector $\begin{bmatrix} 7 \\ 6 \\ 5 \end{bmatrix} \in \mathbb{R}^3$.

(c) Construct an example of a linear transformation whose kernel is the plane $x + 4y - z = 0$ in $\mathbb{R}^4$.

Question 3. For a given matrix $\mathbf{A}_{3\times 4}$ in row-reduced echelon form, what can you say about the possible images for $\mathbf{A}$? Describe all of the possible cases in terms of the rank of $\mathbf{A}$. For each case, provide an example of what $\mathbf{A}$ would look like, and draw a sketch for each.

Question 4. Consider a set of perpendicular vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m \in \mathbb{R}^n$. Show that these vectors are linearly independent. Hint: Form the dot product of $\mathbf{v}_i$ with both sides of the equation

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_i\mathbf{v}_i + \ldots + c_m\mathbf{v}_m = \mathbf{0}.$$

Question 5. Consider the matrix

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 2 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

already in row-reduced echelon form. For which positive integers $j = 1, 2, \ldots, 7$, does there exist a vector $\mathbf{x} \in \ker(\mathbf{A})$ such that the jth component x_j of $\mathbf{x}$ is non-zero, while all of the other components $x_{j+1}, \ldots, x_7$ are 0?

Question 6. Do the same exercise as **Question 1** above, this time noting in what vector space these images and kernels "live":

(a) $\mathbf{A} = \begin{bmatrix} 1 & 1 & 3 \\ 2 & 1 & 4 \end{bmatrix}$, (b) $\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 4 \end{bmatrix}$, (c) $\mathbf{A} = \begin{bmatrix} 1 & 0 & 5 & 3 & 0 \\ 0 & 1 & 4 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$, (d) $\mathbf{A} = \begin{bmatrix} 1 & 1 & 5 & 1 \\ 0 & 1 & 2 & 2 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 4 \end{bmatrix}$.

Question 7. For which value(s) of the constant k do the vectors below form a basis of $\mathbb{R}^4$?

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 2 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 1 \\ 0 \\ 3 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 0 \\ 1 \\ 4 \end{bmatrix}, \quad \begin{bmatrix} 2 \\ 3 \\ 4 \\ k \end{bmatrix}.$$

Question 8. Find a basis of the subspace of $\mathbb{R}^4$ defined by the equation $2x_1 - x_2 + 2x_3 + 4x_4 = 0$.

Question 9. Questions to argue over:

(a) Consider a non-empty subset W of $\mathbb{R}^n$ that is closed under addition of vectors and scalar multiplication. Is W necessarily a subspace of $\mathbb{R}^n$?

(b) Consider the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m \in \mathbb{R}^n$, where $\mathbf{v}_m = \mathbf{0}$. Are these vectors linearly independent? Explain.

(c) Consider $\mathbf{A}_{m \times n}$ and $\mathbf{B}_{n \times m}$, where $\mathbf{AB} = \mathbf{I}_m$. (We say that $\mathbf{A}$ is a *left inverse* of $\mathbf{B}$.) Are the columns of $\mathbf{B}$ linearly independent? How about the columns of $\mathbf{A}$? (Keep in mind that things may be different depending on whether $m < n$, $m = n$, or $m > n$.)

(d) For two subspaces $\mathbf{V}$ and $\mathbf{W}$ of $\mathbb{R}^n$, where $\mathbf{V}$ is contained inside $\mathbf{W}$. Explain why $\dim(\mathbf{V}) \leq \dim(\mathbf{W})$.

(e) For two subspaces $\mathbf{V}$ and $\mathbf{W}$ of $\mathbb{R}^n$, where $\mathbf{V}$ is contained inside $\mathbf{W}$. Explain why if $\dim(\mathbf{V}) = \dim(\mathbf{W})$, then $\mathbf{V} = \mathbf{W}$.

(f) If $\mathbf{V}$ is a subspace of $\mathbb{R}^n$ and $\dim(\mathbf{V}) = n$, explain why $\mathbf{V} = \mathbb{R}^n$.

(g) Can you find a 3×3 matrix $\mathbf{A}_{3\times3}$ such that $im(\mathbf{A}) = \ker(\mathbf{A})$? If yes, produce one. If no, why not?