CHALLENGE PROBLEM SET: COURSE WEEK 3

110.201 LINEAR ALGEBRA
PROFESSOR RICHARD BROWN

Question 1. For the following system

$$\begin{aligned} x + 2y + 3z &= 1 \\ 4x + 5y + 6z &= 4 \\ 7x + 8y + 9z &= k, \end{aligned}$$

where $k \in \mathbb{R}$, do the following:

(a) Write this system as a single matrix equation.

(b) Write this system an as augmented matrix. In this case, put the augmented matrix in reduced-row echelon form. Determine for every value of k the number of solutions of the system.

Question 2. Divide your group into three groups and work the three following problems in parallel on a board. Present your solutions in turn and contrast your solution with the others in this group. Discuss.

(a) Find a polynomial $f(t)$ of degree ≤ 2 of the form $f(t) = a + bt + ct^2$ whose graph goes through the points $(1, p)$, $(2, q)$ and $(3, r)$, where p, q, r are arbitrary constants. Does such a polynomial exist for all values of p, q, r?

(b) Find a polynomial $f(t)$ of degree ≤ 2 of the form $f(t) = a + bt + ct^2$ whose graph goes through the points $(1, 1)$ and $(2, 0)$ such that $\int_1^2 f(t)\, dt = -1$. Will this polynomial still exist if we change the integral value to any other value besides -1?

(c) Find a polynomial $f(t)$ of degree ≤ 2 of the form $f(t) = a + bt + ct^2$ whose graph goes through the points $(1, 1)$ and $(3, 3)$ such that $f'(2) = 3$. Will this polynomial still exist if we change the derivative value to any other value besides 3?

Question 3. Find all vectors in $\mathbb{R}^4$ that are perpendicular to the three vectors

$$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} 1 \\ 9 \\ 9 \\ 7 \end{bmatrix}.$$

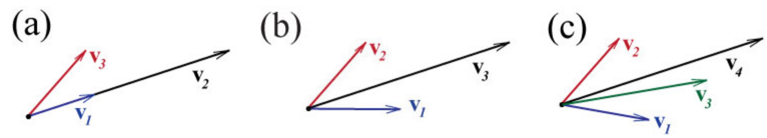
Question 4. Consider a linear system of four equations with three unknowns. We are told that the system has a unique solution. What does the reduced row-echelon form of the coefficient matrix of this system look like? Fully explain your answer.

Question 5. In each case, the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4 \in \mathbb{R}^2$ are shown in the sketch. Arguing geometrically, answer the following: determine how many solutions there are (values for x and y in the first two and x, y and z in the last) for each system given:

(a) How many solutions x and y does the system $x\mathbf{v}_1 + y\mathbf{v}_2 = \mathbf{v}_3$ have?

(b) How many solutions x and y does the system $x\mathbf{v}_1 + y\mathbf{v}_2 = \mathbf{v}_3$ have?

(c) How many solutions x, y and z does the system $x\mathbf{v}_1 + y\mathbf{v}_2 + z\mathbf{v}_3 = \mathbf{v}_4$ have?



Question 6. Again divide into the three groups and again work the three following problems in parallel on a board. Present your solutions in turn and contrast your solution with the others in this group. Discuss.

(a) Let $\mathbf{A}$ be a 4×4 matrix, and let $\mathbf{b}, \mathbf{c} \in \mathbb{R}^4$ be two vectors. We are told that the system $\mathbf{Ax} = \mathbf{b}$ has a unique solution. What can you say about the number of solutions of the system $\mathbf{Ax} = \mathbf{c}$?

(b) Let $\mathbf{A}$ be a 4×4 matrix, and let $\mathbf{b}, \mathbf{c} \in \mathbb{R}^4$ be two vectors. We are told that the system $\mathbf{Ax} = \mathbf{b}$ is inconsistent. What can you say about the number of solutions of the system $\mathbf{Ax} = \mathbf{c}$?

(c) Let $\mathbf{A}$ be a 4×3 matrix, and let $\mathbf{b}, \mathbf{c} \in \mathbb{R}^4$ be two vectors. We are told that the system $\mathbf{Ax} = \mathbf{b}$ has a unique solution. What can you say about the number of solutions of the system $\mathbf{Ax} = \mathbf{c}$?

Question 7. Following Homework Exercise 1.3.49, for an arbitrary positive integer $n \geq 3$, find all solutions $x_1, x_2, \ldots, x_n$ of the simultaneous equations $x_2 = \frac{1}{2}(x_1 + x_3)$, $x_3 = \frac{1}{2}(x_2 + x_4)$, $\ldots$, $x_{n-1} = \frac{1}{2}(x_{n-2} + x_n)$. That is, you are asked to solve the system of equations given by

$$x_k = \frac{1}{2}\left(x_{k-1} + x_{k+1}\right), \quad \text{for} \quad k = 2, 3, \ldots, n-1.$$