## Sample questions for the final

(1) Let $A$ be a $2 \times 2$ matrix with eigenvalues 1 and 4. Let Ker $(A - I) =$ Span$\{\begin{bmatrix} -2 \\ 1 \end{bmatrix}\}$ and Ker $(A - 4I) =$ Span$\{\begin{bmatrix} 3 \\ -1 \end{bmatrix}\}$.

- (a) Is $A$ diagonalizable? If yes, write out the diagonalization, else explain why $A$ is not digaonalizable?
- (b) Find a diagonal matrix $B$ such that $B^2 = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$.
- (c) Use parts 8(a) and 8(b) to find a matrix $X$ such that $X^2 = A$.

(2) Let $A$ be the matrix $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$.

- (a) Find the eigenvalues and eigenspaces of $A$. Write down an orthogonal basis of $\mathbb{R}^2$ consisting of eigenvectors of $A$.
- (b) Find a orthogonal basis of $\mathbb{R}^2$ consisting of eigenvectors of $A$.
- (c) Let $T$ denote the transformation described by $A$. Write down the matrix of $T$ with respect to the new eigenbasis you wrote down in 2(a).
- (d) Explain what the diagonalization of $A$ describes in terms of $T$.

3 Solve the following system of differential equations.

$$\frac{dx_1}{dt} = x_1(t) - 2x_2(t)$$
$$\frac{dx_2}{dt} = 2x_1(t) + x_2(t)$$

Given $x_1(0) = 1$ and $x_2(0) = -1$. What happens to $x_1(t), x_2(t)$ as $t \to \infty$?

(3) Answer the following in short. Give justification for your answers.

- (a) Let $\det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = 6$. Find $\det \begin{bmatrix} a+2d & b+2e & c+2f \\ g & h & i \\ 2d & 2e & 2f \end{bmatrix}$.
- (b) Let $V =$ Span$\{\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ -2 \end{bmatrix}\}$. Find a basis of $V$.
- (c) Give an example of a $3 \times 3$ matrix $A$ with eigenvalues 5, -1 and 3.
- (d) If $A$ is a $3 \times 3$ orthogonal matrix find all possible values of its determinant.
- (e) Let $A^2 = I$. Find Ker $A$.

(4) State true or false with justification.

(a) Let $A$ be a $3\times 3$ matrix. If $Ax = 0$ has infinitely many solutions then the column vectors of $A$ span $\mathbb{R}^3$.
(b) Let $A$ be a $3\times 3$ matrix with a set of eigenvectors spanning $\mathbb{R}^3$. Then $A$ is diagonalizable.
(c) Let $A$ be a $3 \times 3$ matrix with linearly independent column vectors. Then $A$ is diagonalizable.
(d) If $A$ is an invertible $3 \times 3$ matrix then $AB = AC$ implies $B = C$.

(5) State whether the following are subspaces of $\mathbb{R}^3$. Justify your answers.

(a) $\left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \Big/ \begin{array}{rcl} x+y & = & -z \\ 2x-1 & = & y \end{array} \right\}$.

(b) $\left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$.

(6) Write short answers to the following.

(i) Let $\begin{bmatrix} 1 & 3 & -1 \\ 0 & -5 & 2 \\ 2 & -1 & 0 \end{bmatrix}$ be the inverse of $A$ . Find an appropriate matrix $X$ so that $XA = \begin{bmatrix} 1 & 2 \\ 1 & 1 \\ 0 & 3 \end{bmatrix}^T$. Is $X$ invertible? Why or why not?

(ii) $A$ is a diagonalizable $2 \times 2$ matrix with eigenvalues 1 and -1. Show that $A^2 = I$.

(iii) $A$ is a $n\times n$ matrix such that $AA^T = I$. What values can determinant of $A$ take?

(iv) If $\{v_1, v_2, v_3\}$ are linearly independent vectors in $\mathbb{R}^5$ and $v_4 = v_3 - v_2 + v_1$, then is $\{v_1, v_2, v_4\}$ is linearly independent? Why or why not?

(v) If $A$ has eigenvalues 1, 3 and $\frac{2}{3}$, find determinant of $A$.

(vi) If $A$ is a invertible $3 \times 3$ matrix and $v_1, v_2, v_3$ are linearly indpendent vectors in $\mathbb{R}^3$. Show that $Av_1, Av_2, Av_3$ are linearly independent.

(7) State True or False with justification.(*No points for just stating true or false*)

(i) Let $C = AB$ for $4 \times 4$ matrices $A$ and $B$. If $C$ is invertible then $A$ is invertible.

(ii) Let $W$ be a subspace of $\mathbb{R}^4$ and $v$ be a vector in $\mathbb{R}^4$. If $v \in W$ and $v \in W^{\perp}$ then $v = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$.

(iii) Let $V$ be a vector space and $W$ be a subspace of $V$. If Dim $W =$ Dim $V$ then $W = V$.

(iv) If $A$ is a invertible $3 \times 3$ matrix and $B$ and $C$ are $3 \times 3$ matrices , then $AB = AC$ implies $B = C$.