Name ....

Suggested Answers to Practice Exam 2
Math 201

(1) Let $T : I\!R^4 \to I\!R^4$ be defined by a matrix

$$A = \begin{bmatrix} 1 & 0 & -2 & 2 \\ 0 & 1 & -1 & 0 \\ 3 & 2 & 0 & 1 \\ 1 & -1 & 0 & 4 \end{bmatrix}$$

(a) Find the Kernel of T.

Soln: $\text{Ker}(A) = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}$.

(b) Is T invertible? Why or why not?

Soln: Yes T is invertible. This is because if Ker $A = \vec{0}$, then A has full Rank $= 4$. This means A is invertible.

(2a) *3pts* When is a set of vectors $\{v_1, \cdots, v_n\}$ in a vector space V said to be a basis of V?

Soln: A set of vectors $\{v_1, \cdots, v_n\}$ in a vector space V said to be a basis of V if it is linearly independent and $\text{Span}\{v_1, \cdots, v_n\} = V$.

(2b) *5pts.* Let $A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 4 & 1 & 2 \\ -1 & 2 & 1 & 1 \\ 2 & 0 & -1 & 0 \end{bmatrix}$ represent a linear transformation from $T : I\!R^4 \to I\!R^4$. Find a basis for the ImT.

Soln: $\text{Im}(T) = \text{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix} \right\}$. To find the basis we only need to get rid of the linearly dependent vectors. The idea is to find the row echelon form for A to identify the linearly independent columns.

$$\begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 4 & 1 & 2 \\ -1 & 2 & 1 & 1 \\ 2 & 0 & -1 & 0 \end{bmatrix}$$

Do R3 + R1 and R4 -2R1 to get,

$$\begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 4 & 1 & 2 \\ 0 & 4 & 1 & 2 \\ 0 & -4 & -1 & -2 \end{bmatrix}$$

Row operations R3-R2 and R4-R2 imply,

$$\begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 4 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The pivot columns give the linearly independent columns of A which span the column space of A. Therefore, a basis for the column space of A will be $\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 2 \\ 0 \end{bmatrix} \right\}$

(3) *8pts.* Check if the following sets are subspaces of $\mathbb{R}^3$ and $\mathbb{R}^4$ respectively or not. Explain your answers.

(a) $W = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : \begin{matrix} x - y + z = 0 \\ x + 1 - 2z = 0 \end{matrix} \right\}$

Soln: No, W is not a subspace of $\mathbb{R}^3$ and hence not a vectorspace under usual vector addition and scalar multiplication. This is because the vector $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ is not in the set as $0 + 1 - 2(0) \neq 0$.

(b) $V = \left\{ \begin{bmatrix} 0 \\ a+b \\ c \\ c-5a \end{bmatrix} : a, b, c \in \mathbb{R} \right\}$.

Soln: Any vector in V is given by $\begin{bmatrix} 0 \\ a+b \\ c \\ c-5a \end{bmatrix}$ for some real numbers a, b and c. Now,

$$\begin{bmatrix} 0 \\ a+b \\ c \\ c-5a \end{bmatrix} = a \begin{bmatrix} 0 \\ 1 \\ 0 \\ -5 \end{bmatrix} + b \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

Hence, $V = \text{Span}\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ -5 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}$. Thus, we conclude that V is a vector space since it can be written as a span of a set of vectors.

(4) *16pts.* Give short answers to the following.

(a) If A and B are 2×2 matrices such that $AB = 0$ then, either $A = 0$ or $B = 0$.

Soln: False. For example $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

(b) Suppose that A is a 3×3 matrix such that $Ax = x$ for all $x \in \mathbb{R}^3$. Let I_3 be the 3×3 identity matrix. Find $\text{Ker}(A - I_3)$, that is the Kernel of the transformation represented by $A - I_3$.

Soln: We are given that $Ax = x$ for all $x \in \mathbb{R}^3$. This means that $Ax = Ix$ since I is the identity matrix. Then we have that $Ax - Ix = 0$ for all $x \in \mathbb{R}^3$. Finally this means $(A - I)x = 0$ for all $x \in \mathbb{R}^3$.

(c) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the transformation which maps a vector $v \in \mathbb{R}^2$ to its reflection along a line along $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Describe the matrix of this transformation.

Soln: You can use formula to check that T maps $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ to $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ to $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Thus the matrix is given by $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

(d) Is $\left\{ \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ -1 \end{bmatrix} \right\}$ a basis of $\mathbb{R}^3$? Why or why not?

Soln: For the given set to be a basis of $\mathbb{R}^3$ it has to be linearly independent and has to span $\mathbb{R}^3$. Now, to check for linear independence we solve the homogeneous system with these vectors as the columns of the coefficient matrix. The augmented matrix for the system will be

$$\begin{bmatrix} 1 & 3 & 4 & 0 \\ 3 & 2 & 5 & 0 \\ -1 & 0 & -1 & 0 \end{bmatrix}$$

By row operations R2 - 3 R1 and R3 + R1

$$\begin{bmatrix} 1 & 3 & 4 & 0 \\ 0 & -7 & -7 & 0 \\ 0 & 3 & 3 & 0 \end{bmatrix}$$

R3 - 3/7 R2

$$\begin{bmatrix} 1 & 3 & 4 & 0 \\ 0 & -7 & -7 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This implies these vectors are linearly dependent and cannot form a basis of $I\!R^3$.