Partial solutions to Sample questions for the final

Please use these only as a reference, there might be typos and errors.

(1) Let A be a 2×2 matrix with eigenvalues 1 and 4. Let Ker $(A - I) = \text{Span}\{\begin{bmatrix} -2 \\ 1 \end{bmatrix}\}$ and Ker $(A - 4I) = \text{Span}\{\begin{bmatrix} 3 \\ -1 \end{bmatrix}\}$.

(a) Is A diagonalizable? If yes, write out the diagonalization, else explain why A is not digaonalizable?

Hint. Since A is a 2×2 matrix with 2 distinct eigenvalues, A is diagonalizable . The space Ker $(A - I)$ is the eigenspace of $\lambda = 1$ and Ker $(A - 4I)$ is the eigenspace of $\lambda = 4$. So now given the eigenvectors we can diagonalize the matrix., $A = SDS^{-1}$ where S consists of the eigenvectors and D has its diagonal elements as the eigenvalues.

(b) Find a diagonal matrix B such that $B^2 = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$.

Ans. We just need to take square root of the diagonal elements since B has to be diagonal.

(c) Use parts 1(a) and 1(b) to find a matrix X such that $X^2 = A$.

Ans. Now $A = SDS^{-1}$ and $D = B^2$. Substitute to find X.

(2) Let A be the matrix $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$.

(a) Find the eigenvalues and eigenspaces of A.

Ans. Eigenvalues are 1 and 3 with eigenspaces $\text{Span}\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\}, \text{Span}\{\begin{bmatrix} -1 \\ 1 \end{bmatrix}\}$.

(b) Find a orthogonal basis of $\mathbb{R}^2$ consisting of eigenvectors of A.

Ans. $\mathcal{B} = \{\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix}\}$.

(c) Let T denote the transformation described by A. Write down the matrix of T with respect to the new eigenbasis you wrote down in 2(a).

Ans. The matrix of T with respect to $\mathcal{B}$ is given by $\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$

(d) Explain what the diagonalization of A describes in terms of T.

Ans. The equality $A = SDS^{-1}$ describes the relation between the transformation T in standard coordinates and that in $\mathcal{B}$ coordinates.

(3) Solve the following system of differential equations.

$$\frac{dx_1}{dt} = x_1(t) - 2x_2(t)$$
$$\frac{dx_2}{dt} = 2x_1(t) + x_2(t)$$

Given $x_1(0) = 1$ and $x_2(0) = -1$. What happens to $x_1(t), x_2(t)$ as $t \to \infty$?

Ans. We find out that the matrix has eigenvalues $1 \pm 2i$ and therefore,

$$\begin{aligned} x(t) &= c_1 e^{(1+2i)t}\vec{v}_1 + c_2 e^{(1-2i)t}\vec{v}_2 \\ &= e^t(c_1 e^{(2i)t}\vec{v}_1 + c_2 e^{(-2i)t}\vec{v}_2) \end{aligned}$$

As $t \to \infty$, $x(t)$, $e^t \to \infty$. Further given the form of the matrix in the dynamical system, we can say that $x(t)$ takes values in an outward circular spiral. (you need the last bit only if the question asks for a phase potrait.)

(4) Find all solutions in C^∞ to the differential equation

$$f'''(t) + f'(t) = e^t$$

given that $f_p(t) = e^t/2$ satisfies the equation.

Ans. Given the particular solution you only need to find solutions to

$$f'''(t) + f'(t) = 0.$$

Eigenfunction gives solution $\lambda = \pm i$. Then solutions to the differential equation are given by

$$f(t) = c_1 \cos t + c_2 \sin t + e^t/2,$$

for real numbers c_1 and c_2.

(5) Answer the following in short. Give justification for your answers.

(a) Let $\det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = 6$. Find $\det \begin{bmatrix} a+2d & b+2e & c+2f \\ g & h & i \\ 2d & 2e & 2f \end{bmatrix}$.

Ans. $\det \begin{bmatrix} a+2d & b+2e & c+2f \\ g & h & i \\ 2d & 2e & 2f \end{bmatrix} = -12$

(b) Let $V = \text{Span}\{\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ -2 \end{bmatrix}\}$. Find a basis of V.

Ans. We need to find out which ones of these vectors are linearly independent.

$$a_1\begin{bmatrix}1\\1\\-1\end{bmatrix}+a_2\begin{bmatrix}2\\3\\0\end{bmatrix}+a_3\begin{bmatrix}0\\-1\\-2\end{bmatrix} = \vec{0}$$

$$\left[\begin{array}{ccc|c}1&2&0&0\\1&3&-1&0\\-1&0&-2&0\end{array}\right]$$

$$\left[\begin{array}{ccc|c}1&2&0&0\\0&1&-1&0\\0&2&-2&0\end{array}\right]$$

$$\left[\begin{array}{ccc|c}1&2&0&0\\0&1&-1&0\\0&0&0&0\end{array}\right]$$

We can see that the two pivots are 1 and 1. This shows that $\begin{bmatrix}1\\1\\-1\end{bmatrix}$ and $\begin{bmatrix}2\\3\\0\end{bmatrix}$ are linearly independent and span V. Therefore, they form a basis of V.

(c) Give an example of a 3×3 matrix A with eigenvalues 5, -1 and 3.

Ans. $\begin{bmatrix}5&0&0\\0&-1&0\\0&0&3\end{bmatrix}$.

(d) If A is a 3×3 orthogonal matrix find all possible values of its determinant.

Ans. $\det A = \pm 1$

(e) Let $A^2 = I$. Find Ker A.

Ans. A is invertible $\implies$ that Ker $A = \vec{0}$.

(6) State true or false with justification.

(a) Let A be a 3×3 matrix. If $Ax = 0$ has infinitely many solutions then the column vectors of A span $\mathbb{R}^3$.

Ans. False : If $Ax = 0$ has infinitely many solutions than A is not invertible and therefore the column vectors are not linearly independent and cannot span $\mathbb{R}^3$.

(b) Let A be a 3×3 matrix with a set of eigenvectors spanning $\mathbb{R}^3$. Then A is diagonalizable.

Ans. True. If the eigenvectors span $\mathbb{R}^3$ then there exist 3 linearly independent eigenvectors of A and therefore, A is diagonalizable.

(c) Let A be a 3×3 matrix with linearly independent column vectors. Then A is diagonalizable.

Ans. False : For instance $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix}$ is invertible but not diagonalizable.

(d) If A is an invertible 3×3 matrix then $AB = AC$ implies $B = C$.

Ans. True. Multiply both sides by A^{-1}.

(7) State whether the following are subspaces of $\mathbb{R}^3$. Justify your answers.

(a) $\left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \Big/ \begin{array}{rcl} x+y & = & -z \\ 2x-1 & = & y \end{array} \right\}$.

Ans. False: Does not have the 0 vector.

(b) $\left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$.

Ans. True : check to see it has all the right properties.

(8) Write short answers to the following.

(i) Let $\begin{bmatrix} 1 & 3 & -1 \\ 0 & -5 & 2 \\ 2 & -1 & 0 \end{bmatrix}$ be the inverse of A . Find an appropriate matrix X so that $XA = \begin{bmatrix} 1 & 2 \\ 1 & 1 \\ 0 & 3 \end{bmatrix}^T$. Is X invertible? Why or why not?

Ans. $X = \begin{bmatrix} 1 & 2 \\ 1 & 1 \\ 0 & 3 \end{bmatrix}^T A^{-1}$. X cannot be invertible since it is not a square matrix.

(ii) A is a diagonalizable 2×2 matrix with eigenvalues 1 and -1. Show that $A^2 = I$.

Ans. Write down the diagonalization of A.

(iii) A is a $n \times n$ matrix such that $AA^T = I$. What values can determinant of A take?

Ans. Apply determinants to the equation $AA^T = I$.

(iv) If $\{v_1, v_2, v_3\}$ are linearly independent vectors in $\mathbb{R}^5$ and $v_4 = v_3 - v_2 + v_1$, then is $\{v_1, v_2, v_4\}$ is linearly independent? Why or why not?

Ans. Write down equation $a_1v_1 + a_2v_2 + a_3v_4 = \vec{0}$ and substitute for v_4. Use linear independence of v_1, v_2, v_3 to show that $a_1 = a_2 = a_3 = 0$.

(v) If A has eigenvalues 1, 3 and $\frac{2}{3}$, find determinant of A.

Ans. determinant = 2 (product of the eigenvalues).

(vi) If A is a invertible 3×3 matrix and v_1, v_2, v_3 are linearly indpendent vectors in $\mathbb{R}^3$. Show that Av_1, Av_2, Av_3 are linearly independent.

Ans. Similar to part iv

(9) Let

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

(a) Find the orthogonal diagonalization of A.

Ans. Eigenvalues are 1 and 3. The corresponding eigenspaces are Span$\{\begin{bmatrix} 1 \\ -1 \end{bmatrix}\}$ and Span$\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\}$. Normalize these vectors to get the orthogonal diagonalization

$$A = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}^T$$

(b) Does the following equation represent an ellipse or a hyperbola.

$$2x_1^2 + 2x_1x_2 + 2x_2^2 = 1$$

Ans. We can write the above equation as

$$\begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 1$$

$$\begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}^T \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 1$$

Let $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}^T \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

Then we get

$$c_1^2 + 3c_2^2 = 1.$$

Clearly this describes an ellipse with minor, major axis along $\begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}$ and $\begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$.

(10) State True or False with justification.

(i) Let $C = AB$ for 4×4 matrices A and B. If C is invertible then A is invertible.

Ans. Use determinant to observe that determinant of A has to be nonzero and hence A has to be invertible. True!

(ii) Let W be a subspace of $\mathbb{R}^4$ and v be a vector in $\mathbb{R}^4$. If $v \in W$ and $v \in W^\perp$ then $v = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$.

Ans. If $v \in W$ and $v \in W^\perp$ then $v.v = 0$ therefore $v = 0$.

(iii) Let V be a vector space and W be a subspace of V. If Dim $W =$ Dim V then $W = V$.

Ans. If Dim $W =$ Dim $V = k$. Then W has k linearly independent vectors in it which span W. But $W \subset V$ implies these vectors are in V also and are linearly independent and since Dim $V = k$ they have to span V. Therefore $W = V$.

(iv) If A is a invertible 3×3 matrix and B and C are 3×3 matrices , then $AB = AC$ implies $B = C$.

Ans. Multiply by A^{-1}.