Sample questions for the final

Note these do not cover all of the syllabus for the test and are only to be used as a sample.

(1) Let A be a 2×2 matrix with eigenvalues 1 and 4. Let Ker $(A - I) =$ Span$\{\begin{bmatrix} -2 \\ 1 \end{bmatrix}\}$ and Ker $(A - 4I) =$ Span$\{\begin{bmatrix} 3 \\ -1 \end{bmatrix}\}$.

(a) Is A diagonalizable? If yes, write out the diagonalization, else explain why A is not digaonalizable?

(b) Find a diagonal matrix B such that $B^2 = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$.

(c) Use parts 1(a) and 1(b) to find a matrix X such that $X^2 = A$.

(2) Let A be the matrix $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$.

(a) Find the eigenvalues and eigenspaces of A. Write down an orthogonal basis of $\mathbb{R}^2$ consisting of eigenvectors of A.

(b) Find a orthogonal basis of $\mathbb{R}^2$ consisting of eigenvectors of A.

(c) Let T denote the transformation described by A. Write down the matrix of T with respect to the new eigenbasis you wrote down in 2(a).

(d) Explain what the diagonalization of A describes in terms of T.

(3) Solve the following system of differential equations.

$$\frac{dx_1}{dt} = x_1(t) - 2x_2(t)$$
$$\frac{dx_2}{dt} = 2x_1(t) + x_2(t)$$

Given $x_1(0) = 1$ and $x_2(0) = -1$. What happens to $x_1(t), x_2(t)$ as $t \to \infty$?

(4) Find all solutions in C^∞ to the differential equation

$$f'''(t) + f'(t) = e^t$$

given that $f_p(t) = e^t/2$ satisfies the equation.

(5) Answer the following in short. Give justification for your answers.

(a) Let $\det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = 6$. Find $\det \begin{bmatrix} a+2d & b+2e & c+2f \\ g & h & i \\ 2d & 2e & 2f \end{bmatrix}$.

(b) Let $V =$ Span$\{\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ -2 \end{bmatrix}\}$. Find a basis of V.

(c) Give an example of a 3×3 matrix A with eigenvalues 5, -1 and 3.

(d) If A is a 3×3 orthogonal matrix find all possible values of its determinant.
(e) Let $A^2 = I$. Find Ker A.

(6) State true or false with justification.
(a) Let A be a 3×3 matrix. If $Ax = 0$ has infinitely many solutions then the column vectors of A span $\mathbb{R}^3$.
(b) Let A be a 3×3 matrix with a set of eigenvectors spanning $\mathbb{R}^3$. Then A is diagonalizable.
(c) Let A be a 3×3 matrix with linearly independent column vectors. Then A is diagonalizable.
(d) If A is an invertible 3×3 matrix then $AB = AC$ implies $B = C$.

(7) State whether the following are subspaces of $\mathbb{R}^3$. Justify your answers.
(a) $\left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \middle/ \begin{array}{rcl} x+y & = & -z \\ 2x-1 & = & y \end{array} \right\}$.
(b) $\left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$.

(8) Write short answers to the following.

(i) Let $\begin{bmatrix} 1 & 3 & -1 \\ 0 & -5 & 2 \\ 2 & -1 & 0 \end{bmatrix}$ be the inverse of A . Find an appropriate matrix X so that $XA = \begin{bmatrix} 1 & 2 \\ 1 & 1 \\ 0 & 3 \end{bmatrix}^T$. Is X invertible? Why or why not?

(ii) A is a diagonalizable 2×2 matrix with eigenvalues 1 and -1. Show that $A^2 = I$.

(iii) A is a $n \times n$ matrix such that $AA^T = I$. What values can determinant of A take?

(iv) If $\{v_1, v_2, v_3\}$ are linearly independent vectors in $\mathbb{R}^5$ and $v_4 = v_3 - v_2 + v_1$, then is $\{v_1, v_2, v_4\}$ is linearly independent? Why or why not?

(v) If A has eigenvalues 1, 3 and $\frac{2}{3}$, find determinant of A.

(vi) If A is a invertible 3×3 matrix and v_1, v_2, v_3 are linearly indpendent vectors in $\mathbb{R}^3$. Show that Av_1, Av_2, Av_3 are linearly independent.

(9) Let

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

(a) Find the orthogonal diagonalization of A.
(b) Does the following equation represent an ellipse or a hyperbola.

$$2x_1^2 + 2x_1x_2 + 2x_2^2 = 1$$

(10) State True or False with justification.

(i) Let $C = AB$ for 4×4 matrices A and B. If C is invertible then A is invertible.

(ii) Let W be a subspace of $\mathbb{R}^4$ and v be a vector in $\mathbb{R}^4$. If $v \in W$ and $v \in W^{\perp}$ then $v = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$.

(iii) Let V be a vector space and W be a subspace of V. If Dim $W =$ Dim V then $W = V$.

(iv) If A is a invertible 3×3 matrix and B and C are 3×3 matrices , then $AB = AC$ implies $B = C$.