November 11, 2009

MATH 110.201
Partial Solutions to Review Exercises

Note you are expected to fill in the details, this is just to help you check your answers

- Determine whether or not the following sets span $I\!R^2$.

  (a) $\{\begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \end{bmatrix}\}$.

  Ans: Yes, because this is a linearly indepedent set with two vectors and dimension of $I\!R^2$ is 2.

  (b) $\{\begin{bmatrix} -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \end{bmatrix}\}$

  Ans: Yes, because it has two linearly independent vectors.

- Give short answers to the following.

  (a) Let $\{v_1, v_2, v_3\}$ span a vector space V and $v_3 \in \text{Span}\{v_1, v_2\}$. Show that $\text{Span}\{v_1, v_2\} = V$.

  Ans: Show that you can write any $v \in V$ as a linear combination of v_1 and v_2.

  (b) If $Ax = 0$ for all $x \in I\!R^3$. Find Ker(A).

  Ans: It should be $I\!R^3$. Explain why?

  (c) If A is a 4×6 matrix with $dim\,\text{Ker}(A) = 2$. How many pivots does A have ? Why?

  Ans: Use Rank Nulity theorem.

- Let V be a vector space of dimension 4 and W be a subspace of V.

  (a) What are the minimum and maximum dimensions W can have? Why?

  Ans: Dimension of a subspace is always less than that of the vectorspace in which it is contained. What is the minimum dimension any space can have?

  (b) Let $\{w_1, w_2, w_3\}$ span W and $w_4 \in V$ be such that $\{w_1, w_2, w_3, w_4\}$ is linearly independent. What is the dimension of W? State the properties of linearly independent sets you are using.

  Ans: 3. Subset of a linearly independent set is linearly independent. How can you apply this here?

- Let $T : I\!R^3 \to I\!R^3$ be the linear transformation defined as

$$T(\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}) = \begin{bmatrix} y_1 + y_2 \\ y_1 - y_2 \\ y_3 \end{bmatrix}$$

(a) Find the Kernel of T.

Ans: Solve the system to see that Kernel$T = \{\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\}$.

(b) Is T invertible? Explain why or why not?
Ans. Yes. This should follow from previous problem.

- Is $W = \{\begin{bmatrix} a-b \\ b \\ 0 \end{bmatrix} : a, b \in I\!R\}$ a subspace of $I\!R^3$ (This is the set of vectors in $I\!R^4$ of the form $\begin{bmatrix} a-b \\ b \\ 0 \end{bmatrix}$ for all possible real values of a and b) ?

(a) Show that W is a subspace of $I\!R^3$.
Ans. You can check all the three conditions,

  * if the $\vec{0}$ is of the form $\begin{bmatrix} a-b \\ b \\ 0 \end{bmatrix}$,
  * sum of two vectors of this form $\begin{bmatrix} a-b \\ b \\ 0 \end{bmatrix}$ and $\begin{bmatrix} c-d \\ d \\ 0 \end{bmatrix}$ is of this form again and
  * the scalar multiple of $\begin{bmatrix} a-b \\ b \\ 0 \end{bmatrix}$ is of the same form.

Alternately, you can write W as a span of vectors and use the theorem that span of vectors is subspace.

(b) Find a basis for W.
Ans. Note $W = \text{Span}\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}\}$. Both these vectors are linearly independent because they are not scalar multiples of each other. Therefore, $\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}\}$ is a basis of W.

- State whether the following statements are true or false. If true, explain your answer. If false, give an example for which the statement is false.

(a) Let W be a subspace of $I\!R^4$. If $W = \text{Span}\{\vec{w}_1, \cdots, \vec{w}_k\}$ for vectors $\vec{w}_1, \cdots, \vec{w}_k$ in $I\!R^4$ and the dimension of W is 3 then $k = 3$.

Ans. False, $\{w_1, \cdots, w_k\}$ can be linearly dependent, give an example where the statement is not true.

(b) If A is a orthogonal 3×3 matrix then $\det A > 0$.

Ans. False, give example.

- Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation defined as

$$T\left(\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}\right) = \begin{bmatrix} y_1 + y_2 \\ y_1 - y_3 \\ 0 \end{bmatrix}$$

(a) Find a basis of the Image of T.

Soln: The Image of T is the set of vectors in $\mathbb{R}^3$ which are of the form $T(\vec{y})$ for some $\vec{y} \in \mathbb{R}^3$.
Then

$$\begin{aligned} T\left(\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}\right) &= \begin{bmatrix} y_1 + y_2 \\ y_1 - y_3 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} y_1 \\ y_1 \\ 0 \end{bmatrix} + \begin{bmatrix} y_2 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -y_3 \\ 0 \end{bmatrix} \\ &= y_1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + y_2 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + y_3 \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} \end{aligned}$$

Then Image of $T = \text{Span}\{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}\}$

Clearly $\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}\}$ is a linearly independent set and $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$.

Therefore, the basis for image of $T = \{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}\}$.

(b) What is the Rank of the matrix defining the linear transformation T? Explain your answer.

Soln: The matrix representing T is $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}$. This will have Rank 2 since the dimension of Image T = Rank of the matrix A.

- Let $W = \{ \begin{bmatrix} a \\ b \\ c \end{bmatrix} : a + b + c = 0 \}$ (that is, the set of all vectors $\begin{bmatrix} a \\ b \\ c \end{bmatrix} \in I\!\!R^3$ satisfying the equation $a + b + c = 0$).

(a) Show that W is a subspace of $I\!\!R^3$.

Soln: The subspace W is the set of solutions to the matrix equation

$$[111] \begin{bmatrix} a \\ b \\ c \end{bmatrix} = [0]$$

Then we have one equation in three unknowns. We have infinitely many choices for b and c. Further

$$a = -b - c.$$

Therefore, vectors in W are of the form

$$\begin{aligned} \begin{bmatrix} a \\ b \\ c \end{bmatrix} &= \begin{bmatrix} -b-c \\ b \\ c \end{bmatrix} \\ &= \begin{bmatrix} -b \\ b \\ 0 \end{bmatrix} + \begin{bmatrix} -c \\ 0 \\ c \end{bmatrix} \\ &= b \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + c \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \end{aligned}$$

Therefore, $W = \text{Span}\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \}$

Alternatively,

* We see that $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \in W$ since $0 + 0 + 0 = 0$.

$*$ For any two vectors $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ and $\begin{bmatrix} e \\ f \\ g \end{bmatrix}$ in W. We know that $a+b+c=0$ and $e+f+g=0$. Then $\begin{bmatrix} a \\ b \\ c \end{bmatrix} + \begin{bmatrix} e \\ f \\ g \end{bmatrix} = \begin{bmatrix} a+e \\ b+f \\ c+g \end{bmatrix}$. But $a+b+c+e+f+g = 0+0 = 0$.

$*$ For any $k \in I\!R$ and $\begin{bmatrix} a \\ b \\ c \end{bmatrix} \in W$ we have that $a+b+c=0$. Then $k(a+b+c) = ka+kb+kc = 0 \implies \begin{bmatrix} ka \\ kb \\ kc \end{bmatrix} \in W$ for all $k \in I\!R$.

Then W is a subspace.

(b) Find a basis of W.

Soln: Clearly, no scalar multiple of $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ can be equal to $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$.

The basis of W is $\{\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\}$ since they span the set and are linearly independent.

- Let $\{\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}\}$ be a basis of a subspace W of $I\!R^4$.

(a) Find an orthogonal basis $\{\vec{u}_1, \vec{u}_2\}$ of W. Show work.

Ans. Use Gram Schmidt to obtain an orthogonal basis $\{\begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix}, \begin{bmatrix} -1/2 \\ 1/2 \\ 1/2 \\ -1/2 \end{bmatrix}\}$ of W.

(b) Let $D = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 1 \\ 1 & 0 \end{bmatrix}$ and $Q = [\vec{u}_1 \vec{u}_2]$. Find an upper triangular 2×2 matrix R so that $D = QR$. Show work.

Ans. Write down column vectors in D in terms of $\vec{u}_1$ and $\vec{u}_2$ to get that $R = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}$.

- Let $\mathbb{C}$ be the set of all complex numbers, that is, $\mathbb{C} = \{a + bi : a, b \in I\!R\}$. Let $T : \mathbb{C} \to \mathbb{C}$ be the linear transformation defined as $T(a+bi) = -b+ai$
  (a) Find the matrix B of the transformation T with respect to the basis $\mathcal{B} = \{1 + i, 1 - i\}$. Show work.

Ans. Evaluate T at the basis elements and write in $\mathcal{B}$-coordinates.

$$B = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

(b) Find the matrix A of the transformation T with respect to the basis $\mathcal{B}' = \{1, i\}$. Show work.

Ans. Repeat the same procedure with $\mathcal{B}'$ coordinates.

(c) Find a matrix S such that $A = SBS^{-1}$. Show work.

Ans. Note if you find the change of basis matrix S from $\mathcal{B}$-coordinates to $\mathcal{B}'$ coordinates, then it has the correct property.

- What is the dimension of the vector space of all 1×3 matrices $\mathcal{M}_{1\times 3}$? Explain your answer.

Ans. Dimension is 3. Write down a basis to explain your answer.

- Let $\mathcal{D}$ denote the space of differentiable functions from $I\!R \to I\!R$. Is the function $<,>: \mathcal{D} \times \mathcal{D} \to I\!R$ defined as

$$< f, g >= f(0)g'(0) + f'(0)g(0)$$

an inner product on $\mathcal{D}$?

Ans. False. What happens when $f(x) = x - 1$ and you take its inner product with itself?

- Let $V = \text{Span}\{\begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ -2 \end{bmatrix}\}$. Find the dimension of V. Explain your answer.

Ans. Write down the matrix with vectors in $\{\begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ -2 \end{bmatrix}\}$ as the column vectors. Use the pivot elements to pick out which vectors are linearly independent.

- Let $A = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}$.
  (a) Find the eigenvalues of A.

Ans. Eigenvalues are 2 and 0.

(b) Find the eigenspaces corresponding to eigenvalues of A.

Ans. Eigenspace for 2 is Span$\{\{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\}$. Eigenspace for eigenvalue 0 is Span$\{\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}\}$.