Dynamically speaking

The lecture of Monday 11/24 was not quite finished. We finish it here. The moral of the story will be: *Diagonalization gives us* **power**.

We were considering the simple *dynamical system*

$$\vec{x}(t+1) = A\vec{x}(t), \quad t = 0, 1, 2, ... \tag{1}$$

with $A \in \mathbb{R}^{m\times m}$; and $\vec{x}(t)$ is an $\mathbb{R}^m$-valued function defined on the non-negative integers, the so-called *state vector* of the system, which is assumed to satisfy the recursion relation (1). I feel like changing the symbol from t to n, to make it look more like Calculus II. (1) is then

$$\vec{x}(n+1) = A\vec{x}(n). \tag{1a}$$

Most, though not all, matrices are diagonalizable over $\mathbb{C}$. (Some very simple matrics are not diagonalizable; remembering and appreciating some will add to your understanding of diagonalization.) We proceed on the assumption that A *is* diagonalizable. The outcome is then not so complicated, as we'll see. This will make it transparent how to determine whether (1) is asymptotically stable. We allow $\mathbb{C}$ (the complex numbers) to be the scalars, so every polynomial splits, but the eigenvalues could be complex numbers. A's being diagonalizable means that there is a (complex) eigenbasis for A.

Let $\lambda_1, ..., \lambda_m$ be the eigenvalues of A (listed multiply for multiple eigenvalues), and let $\vec{v}_1, ...\vec{v}_m$ be corresponding eigenvectors, so $A\vec{v}_j = \lambda_j\vec{v}_j$. By the usual process, we obtain

$$D = S^{-1}AS, \qquad \text{with} \qquad S = [\,\vec{v}_1\,|\,...\,|\,\vec{v}_m\,], \tag{2}$$

the matrix with the $\vec{v}_j$'s as column vectors, and D is the diagonal matrix with diagonal entries the λ_j's (in order). Solving (2) for A, we obtain the formula

$$A = SDS^{-1}. \tag{3}$$

Inserting this into (1a) gives $\vec{x}(n+1) = SDS^{-1}\vec{x}(n)$, equivalently

$$S^{-1}\vec{x}(n+1) = DS^{-1}\vec{x}(n). \tag{4}$$

Changing variables to suit the situation, we let

$$\vec{y} = S^{-1}\vec{x},$$

which is just a linear change of variables (from $\vec{x}$ to $\vec{y}$). The equation (4), expressed in terms of $\vec{y}$, has a very simple form:

$$\vec{y}(n+1) = D\vec{y}(n),$$

or written out for each j,

$$y_j(n+1) = \lambda_j y_j(n); \tag{5}$$

diagonalization serves to *uncouple* the variables. We see that (5) admits the explicit (i.e., non-recursive) formula

$$y_j(n) = \lambda_j^n y_j(0). \tag{6}$$

This implies that $\vec{y}(n) \to 0$ *as* $n \to \infty$ *for all initial states if and only if* $|\lambda_j| < 1$ *for all* j.

From (4) we quickly deduce that

$$\vec{x}(n) = S\vec{y}(n) \tag{7}$$

We now put the formulas (6) into (7) to solve the problem. Note that (7) says that each $x_i(n)$ is a specific linear combination of the y_j's—you know,

$$x_i = \sum_{j=1}^{m} s_{ij} y_j. \tag{8}$$

Can you see why $\vec{x}(n) \to 0$ as $n \to \infty$ for all initial states $\vec{x}(0)$ if and only if $\vec{y}(n) \to 0$ as $n \to \infty$ for all initial states? It may not be obvious, but it is true, because S is invertible. (Surely you can see this when $m = 1$, for starters. Maybe consider the case $m = 2$ next, and if you succeed, that may suggest the general pattern.) I will not address this here. We obtain that $\vec{x}(n) \to 0$ *as* $n \to \infty$ *for all initial states if and only if* $|\lambda_j| < 1$ *for all* j.

In sum, asymptotic stability gets decided by the eigenvalues of A. Better living through diagonalization!