Basis for a span

The issue I wish to address here is the question: *Given a set of ℓ vectors in $\mathbb{R}^n$*

$$S = \{\vec{v}_1, \vec{v}_2, ..., \vec{v}_\ell\},$$

find a basis of Span(S). One is asking, in effect, to discard unnecessary (i.e., redundant) vectors.

There are three subquestions, which will be answered in turn:

a) *if you want to use only a subset of the given set of vectors,* there's a rule that may seem like magic, or even nonsense:

1. Make an $n \times \ell$ matrix A with S as column vectors.
2. Reduce A to the reduced row-echelon matrix $rref(A)$.
3. Take the columns of A corresponding to those columns of $rref(A)$ where the leading 1's lie. These vectors give a basis of Span(S).

The mystery in the above lies in the fact that row operations usually *change* the columns of a matrix. (We will proceed differently in (b) below, where we convert our column vectors to row vectors, and then convert back at the end.) A decent way to view this is that row reduction is accomplished by the left-multiplication $A \mapsto EA$, where E is invertible. The reverse process goes likewise, namely by left-multiplication by E^{-1}. (We've touched upon this; see Ch. 2.4, #50–53.) Can you see the process and its inverse take linearly independent sets to linearly independent sets, spans to spans?[1] We then note that im($rref(A)$) has an obvious basis, namely the columns with the leading 1's. Applying E^{-1} to them justifies step #3.

b) *if you are willing to take any basis;*

Lay the (column) vectors in S as the *rows* of an $\ell \times n$ matrix; call it B. Do row operations on B to get it into the reduced row-echelon matrix $rref(B)$. The span of the *rows* of $rref(B)$ and those of B coincide. The non-zero rows, set as column vectors, form a basis for Span(S).

c) *if you want to get a basis for the image of an $n \times \ell$ matrix A.*

This is easy now. Take S to be the set of column vectors of A, and do your favorite of a) or b).

[1] That was easy to say, but let's state it precisely. Let B be an invertible $n \times n$ matrix. Then a set of vectors $S = \{\vec{v}_1, \vec{v}_2, ..., \vec{v}_\ell\}$ in $\mathbb{R}^n$ is linearly independent if and only if $\{B(\vec{v}_1), B(\vec{v}_2), ..., B(\vec{v}_\ell)\}$ is linearly independent. Likewise, a set of vectors S spans a subspace $W \subseteq \mathbb{R}^n$ if and only if $\{B(\vec{v}_1), B(\vec{v}_2), ..., B(\vec{v}_\ell)\}$ spans $B(W)$ (the image of W under B).