Relection on/and change of basis

1. Let $\mathcal{B} = \{\vec{v}_1, ..., \vec{v}_n\}$ be a basis of $\mathbb{R}^n$, and $T : \mathbb{R}^n \to \mathbb{R}^n$ a linear transformation ($T = T_A$ for some $n \times n$ matrix A). There was a loose end in lecture, namely getting the formula for $B = [T]_{\mathcal{B}}$ in terms of A and $\mathcal{B}$. The matrix B is the one for which

$$[T(\vec{x})]_{\mathcal{B}} = B[\vec{x}]_{\mathcal{B}}.$$

For S the matrix with the $\vec{v}_i$'s as column vectors (listed in the order given), we have $[\vec{x}]_{\mathcal{B}} = S^{-1}\vec{x}$ (and equivalently $\vec{x} = S[\vec{x}]_{\mathcal{B}}$) for all $\vec{x} \in \mathbb{R}^n$. Thus, for all $\vec{x}$

$$[T_A(\vec{x})]_{\mathcal{B}} = S^{-1}A\vec{x} = S^{-1}AS[\vec{x}]_{\mathcal{B}}, \qquad \text{so } B = S^{-1}AS.$$

2. Earlier in the semester, we had a version of the following problem: *Let L be the line in $\mathbb{R}^2$ that makes a directed angle θ with the positive x_1-axis. Show that reflection T in L is a linear transformation (and determine its matrix).*

One approach was to change coordinates so that L "became" the x_1-axis. In this case the line perpendicular to L also plays a role, so we make that the x_2-axis. In terms of the new axes, the matrix of T is quite clearly

$$B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

We want to give the matrix of T in old coordinates.

In terms of the notation from lecture, we take $\mathcal{B}$ to be $\{\vec{u}, \vec{w}\}$, with

$$(*) \qquad \vec{u} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} \qquad \vec{w} = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}.$$

The matrix S (inverse coordinate mapping) is $S = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$. The matrix A ($T = T_A$ for the standard basis) is the one that satisfies $S^{-1}AS = B$ (why?). Solve for A:

$$(1) \qquad A = SBS^{-1} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

Earlier, we saw another formula for T:

$$(2) \qquad T(\vec{x}) = (2\vec{u} \bullet \vec{x})\vec{u} - \vec{x},$$

where $\vec{u}$ is either unit vector in the direction of L; use the one in (*). Refer to Ch. 2.2 if necessary. Check that (1) and (2) agree (they must, of course). Some standard trig identities enter.