Correction

While taking a shower yesterday, I realized that I said something blatantly incorrect in Friday's lecture.

At issue was orthogonal projection onto an m-dimensional subspace V of $\mathbb{R}^n$. I asserted that the formula for the (standard basis) matrix of orthogonal projection is QQ^T, where Q is the $n \times m$ matrix whose columns are given by any basis of V. This was questioned by a student: does it have to be an *orthonormal* basis? That student was right.

A quick reason why it is wrong for general bases is to note that if one multiplies all vectors in the basis by a scalar, say 2, both Q and Q^T get multiplied by 2, so QQ^T gets multiplied by 4. Thus, one does not get the same matrix for the two bases. We can see why the formula *is* correct when an orthonormal basis is used (see below). Believe it or not, I went through that earlier in the week!

Let $\mathfrak{B}_1 = \{\vec{v}_1, ..., \vec{v}_m\}$ be an orthonormal basis of V, and use an orthonormal basis $\mathfrak{B}_2$ of $V^\perp$ to make a basis $\mathfrak{B} = \mathfrak{B}_1 \cup \mathfrak{B}_2$ of $\mathbb{R}^n$. The rows of Q^T are the elements of $\mathfrak{B}_1$, laid down as row vectors. From the interpretation of matrix multiplication in terms of dot product and the orthogonality conditions on $\mathfrak{B}$, we get that $Q^T\vec{v}_i = \vec{e}_i \in \mathbb{R}^m$; and $Q^T\vec{w} = \vec{0}$ when $\vec{w} \in \mathfrak{B}_2$ (so $QQ^T\vec{w} = \vec{0}$). On the other hand, note that $QQ^T\vec{v}_i = Q\vec{e}_i$ equals the ith column vector of Q, namely $\vec{v}_i$. In other words, QQ^T does what projection onto V does to the basis $\mathfrak{B}$, so the two linear transformations are equal. *Can you see how that compares to the SBS^{-1} formula from the lecture on 10/22? We used the same sort of basis.*

Taking $V = \mathbb{R}^n$ (so $V^\perp = \{\vec{0}\}$ and $\mathfrak{B}_2$ is empty), one recovers the fact that for an orthogonal matrix Q, $QQ^T = I$.