Lest we've forgotten

I said in lecture today that the solution set of $A\vec{x} = \vec{0}$ (a *homogeneous* system because of the $\vec{0}$ on the right-hand side) is *given by our elimination process* as the span of a specific set of vectors. Again, for given vectors $\vec{v}_1, ..., \vec{v}_\ell$, the *span* of these vectors is the set of all linear combinations of these vectors, namely all vectors that can be written in the form

$$t_1\vec{v}_1 + ... + t_\ell\vec{v}_\ell,$$

where $t_1, ..., t_\ell$ are scalars (i.e., $\in \mathbb{R}$).

Let's illustrate the above by taking a particular linear system. We may as well take the coefficient matrix to be a reduced row-echelon matrix. *WHY?* Take as the matrix A

$$A = \begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

Then $A\vec{x} = \vec{0}$ has solution: $x_1 = -2t$, $x_2 = t_1$ (parameter for the free variable x_2), $x_3 = 0$, $x_4 = t_2$ (parameter for the free variable x_4). In our preferred vector form, that's

$$\vec{x} = t_1 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t_2 \begin{bmatrix} -4 \\ 0 \\ 0 \\ 1 \end{bmatrix},$$

where t_1 and t_2 are arbitrary real numbers. The above **is** the span of those two vectors, giving the solution set to $A\vec{x} = \vec{0}$, which we are calling $\ker A$. Thinking of what the example represents, we can see that the kernel of *any* matrix is equal to the span of a finite number of vectors (possibly zero), namely the number of free variables.

As far as seeing that the image of a matrix, namely the span of its column vectors, is the solution set of some system of linear equations, we will address that later. Soon, but later!