

Reflection on change of matrix

Let A be an $n \times n$ matrix. The words *change of matrix* in the title could refer to what happens when you describe the linear transformation T_A with respect to some basis \mathfrak{B} other than the standard basis \mathfrak{A} . You should first appreciate the statement

$$(1) \quad [T_A]_{\mathfrak{A}} = A,$$

and that it says how T_A acts in terms of \mathfrak{A} -coordinates.

Next, we understand that the \mathfrak{B} -matrix $[T_A]_{\mathfrak{B}}$ can be described in two ways:

a) The matrix $[T_A]_{\mathfrak{B}}$ is the one and only matrix that satisfies:

$$(2) \quad [A\vec{v}]_{\mathfrak{B}} = [T_A(\vec{v})]_{\mathfrak{B}} = [T_A]_{\mathfrak{B}}[\vec{v}]_{\mathfrak{B}}.$$

Once we accept that all bases of \mathbb{R}^n are equal in the eyes of the law, (2) has the same content as (1). It coincides with (1) when $\mathfrak{B} = \mathfrak{A}$, and it says how T_A acts in terms of \mathfrak{B} -coordinates.

b) Since we are presumed to be given the matrix A , there is the well-known formula that minimizes the brain pain:

$$(3) \quad [T_A]_{\mathfrak{B}} = S^{-1}[T_A]_{\mathfrak{A}}S = S^{-1}AS,$$

where S is the matrix whose column vectors are the vectors in \mathfrak{B} .

Note that (3) says, *once you know the vectors in \mathfrak{B} , and how to multiply and invert $n \times n$ matrices, computing the \mathfrak{B} -matrix of T_A is completely mechanical*. A good linear algebra student will understand a) as well.

However, the above is not what I had in mind. I want to point out some very simple maneuvers concerning matrices and their eigenvalues.

i) λ is an eigenvalue of A if and only if $\lambda + c$ is an eigenvalue of $A + cI$. Indeed we have the following elementary calculation

$$(4) \quad A\vec{v} = \lambda\vec{v} \quad \text{if and only if} \quad (A + cI)\vec{v} = (\lambda + c)\vec{v},$$

from which it follows that there is even an *equality* of eigenspaces

$$(5) \quad \ker(A - \lambda I) = E_{\lambda}(A) = E_{\lambda+c}(A + cI).$$

ii) Similarly,

$$(6) \quad E_{\lambda}(A) = E_{k\lambda}(kA).$$