A tale of two bases

Let's start with a tale of *one* basis of $\mathbb{R}^n$. Get down to the bottom of things: let $n = 2$. I hope you see that $n = 1$ is too easy for most things, **but not here!** So **let** $n = 1$.

If I told you that we had a Calc 0 function $f : \mathbb{R}^1 \to \mathbb{R}^1$, and that $f(1) = 2$; determine f. You'd think I was toying with you. You'd say, there are infinitely many such functions, e.g., $f(x) = 2x^k$ for every $k > 0$. And that's absolutely correct. But if I added that f is linear, the only answer would be $f(x) = 2x$.

The message you should have picked up by now is the notion of *linearity*, as I put it. That's Fact 1.3.9 from our book. It follows that *a linear mapping is completely determined by its value on any one basis.* How? Let $\mathcal{B} = \{\vec{v}_1, ..., \vec{v}_n\}$ be a basis of $\mathbb{R}^n$. Write $\vec{v} = c_1\vec{v}_1 + ... + c_n\vec{v}_n$. The c_i's are determined uniquely by $\vec{v}$ (that's what a basis is all about), then

$$(*) \qquad T(\vec{v}) = c_1 T(\vec{v}_1) + ... + c_n T(\vec{v}_n).$$

There are simple rules for working with linear transformations. **Never forget that!** Admittedly, the task of finding the actual c_i's for a given vector $\vec{v}$ (and given $\mathcal{B}$, of course) involves solving a certain system of linear equations, but we should know how to do that by now.

Since $\{1\}$ is a basis of $\mathbb{R}^1$, meaning that every number x is a multiple of 1, linearity gives $f(x) = f(x \cdot 1) = xf(1)$; $f(1) = 2$ thereby determines $f(x) = 2x$.

Let's move onward to $\mathbb{R}^2$ and functions of two variables. Suppose I have a secret function $f : \mathbb{R}^2 \to \mathbb{R}$ and I told you (say) that $f(1, 1) = 2$ and $f(1, -2) = 3$. Tell me my function or you die! That would reduce attendence in lecture drastically. But if I told you the wonderful feature that f is linear, there is sufficient information; there is exactly one such linear f. Why? Because $\mathcal{B} = \{(\vec{e}_1 + \vec{e}_2), (\vec{e}_1 - 2\vec{e}_2)\}$ is a *basis* for $\mathbb{R}^2$ (explain). Linearity, in its version (*), tells you how to deduce the value of f at every vector of $\mathbb{R}^2$. *[Do it.]*

The above contains the essential issues of Linear Algebra, and we have not gotten beyond $\mathbb{R}^2$ as domain and $\mathbb{R}^1$ as codomain!

Here comes the tale of two bases. *From A to Z*: A is a student, Z is a mathematics instructor. [The comments in brackets are not spoken, and they probably wouldn't occur at JHU.]

– Z: Which vector in $\mathbb{R}^2$ has coordinates 1,0?

– A: [$\vec{e}_1$ of course.]

Don't you have to specify a basis?

– Z: Yes, of course. The basis I have in mind is $\mathcal{B} = \{(\vec{e}_1 + \vec{e}_2), (\vec{e}_1 - 2\vec{e}_2)\}$.

– A: Why that one?

– Z: [Do you need a reason?]

It will become clear later on. I have something up my sleeve.

– A: OK. The answer is the first element of your [stupid] basis, namely $\vec{e}_1 + \vec{e}_2$.

– Z: Good, now you're getting serious. Next, tell me the matrix T of the coordinate mapping <u>with respect to the basis $\mathcal{B}$</u>.

– A: OK. Isn't that the one where you put the vectors of $\mathcal{B}$ in order as the columns of T:

(*)
$$T = \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix}$$

[*#*!] Or did I give T^{-1}? ....

– Z: Stay calm. The matrix you gave takes $\vec{e}_1$ to $\vec{e}_1 + \vec{e}_2$, and $\vec{e}_2$ to $\vec{e}_1 - 2\vec{e}_2$. That's backwards, right? The $\mathcal{B}$-coordinate map takes any vector in $\mathbb{R}^2$ to the coefficients of the vectors in the ordered basis under consideration. You really want $\vec{e}_1 + \vec{e}_2$ to map to $\vec{e}_1$, etc. So you have written down the inverse matrix. Can you invert a 2×2 matrix?

– A: Do you take me for a dummy?

– Z. [No comment.] Of course not. Do you see the point of linearity? If you know you have a linear transformation of $\mathbb{R}^m$ (into $\mathbb{R}^n$), it is completely specified by its value on m linearly independent vectors of $\mathbb{R}^m$. Just take linear combos and invoke linearity. (Conceptually, you can think that without carrying it out in numbers. Do you <u>really</u> want to carry that out in (say) $\mathbb{R}^{30}$?)

– A: I think I'm getting the point.

– Z. Great! Now, let's determine the matrix of T_A, where A is the 2×2 matrix:

$$A = \tfrac{1}{3} \begin{bmatrix} 13 & 2 \\ 4 & 11 \end{bmatrix}$$

– A: It's A, of course.

– Z: That's for the standard basis. I've been insisting on my basis $\mathcal{B}$. You'll see why.

We go straight to the ending. The matrix is

$$B = \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix}$$

As the theory produces unambiguously, $B = TAT^{-1}$ ($S^{-1}AS$, with $S = T^{-1}$). *[Check it out.]* The punch line: that *mess* of a matrix A is describing a transformation that stretches by a factor of 5 in one direction (of $\vec{v}_1 = \vec{e}_1 + \vec{e}_2$), and by a factor of 3 in another ($\vec{v}_2 = \vec{e}_1 - 2\vec{e}_2$). So what if these directions aren't perpendicular! Isn't that a clearer picture of what T_A does than you get by just looking at A?

There is a procedure for making transformations look simpler, and it comes later in the course.