Linear Algebra

I want to comment some old themes. Before doing that, you should know that the material of Chapters 3 and 4 **always** causes trouble with students (and I don't mean only the ones who had me as an instructor). The key to understanding this stuff has to come from you. It cannot be resolved in the lectures, mine or anyone else's, I believe, without distorting the course syllabus. The material is clearly presented in the primary source (the textbook) after all, believe it or not.

1. *A linear transformation $T : V \to W$ is determined by its values on any one basis of V.* This is a jazzed up version of the less abstract statement for linear transformations $T : \mathbb{R}^m \to \mathbb{R}^n$, which are given by an $n \times m$ matrix. The matrix (so the entire transformation) is determined by its values on the standard basis of $\mathbb{R}^m$ (the domain); the rest of the values of T_A are thereby determined. We saw this explicitly in that the i-th column vector of an $n \times m$ matrix A is $A\vec{e}_i$ (the value of T_A at $\vec{e}_i$). If that isn't clear by now, ...

All we have done in Ch. 4 is to extend this to cover arbitrary linear spaces and arbitrary bases. Declare where a basis of V is to get mapped, and the rest of the domain follows the instructions given by linearity. *There are other ways of specifying a function than giving a formula for it* (Calc. 0).

Said another way, suppose you wanted to tell me about an exciting mapping from $\mathbb{R}^2$ to $\mathbb{R}^{35}$. Make it a linear mapping; call it T. Well, I need to hear something about T, for all I know otherwise is that $T(\begin{bmatrix} 0 \\ 0 \end{bmatrix}) = \vec{0} \in \mathbb{R}^{35}$. Then you say, trying to be helpful, "I'll give you a table of values:"

$$T(\begin{bmatrix} 1 \\ 0 \end{bmatrix}) = v_1 \text{ (something);} \quad \text{continue } T(\begin{bmatrix} 1 \\ 1 \end{bmatrix}) = v_2, \quad T(\begin{bmatrix} 1 \\ 2 \end{bmatrix}) = v_3, \ldots.$$

At some point, I stop tuning out your flood of data, and exclaim, "Look, you've already mentioned a basis of $\mathbb{R}^2$ (take any two of the vectors in $\mathbb{R}^2$ given above) and gave me the corresponding values of T. I know what *linear* means, and I can take it from there to compute any other value of T. (For instance, why must we have $v_3 = -v_1 + 2v_2$ above?) I can even give you a formula for T—if and when I need to."

2. *The reason for coordinates.* Let's start with the content of coordinates on a line. Referring back to the document "Coordinates," we have that coordinates on a line are just the choice of a positive direction and scale. That's the same as finding a basis for $\mathbb{R}^1$. **Note that all vector addition is accomplished by adding the coordinates, whatever scale you use; likewise scalar multiplication. You just have to remember which scale it is.**

If you had any linear space V of dimension one, you would select a non-zero element v, and take it as basis $\mathfrak{B}$. Then the only thing you need to talk about in dealing with that space is: for each $f \in V$, $f = cv$ for a uniquely determined scalar c; those scalars are all that we need to do to specify *all* vectors in V. Then I can use those c's to do all linear operations in V. The choice of v determines what you call "c," but it makes no difference in the outcome.

The same goes in higher-dimensional linear spaces. We must use several numbers as coordinates, but after doing that, linear algebra is the same as in $\mathbb{R}^n$. Linear transformations are given by matrices, like in the good old days (Chs. 2 and 3).