Coordinates from Ch. 3

This is not really so different from coordinates in general linear spaces, but I'll stay in $\mathbb{R}^n$ here.

1. *Coordinates on a line.* With 0 marked, the only issue is where to put the 1. **Anywhere but 0.**

Suppose we are plotting distance in *feet.* The 1 means one foot in the positive direction (pick one). The selected "unit" vector is the vector from 0 to 1. Though it may seem arbitrary, we'll take this vector to give the standard basis.

Suppose you want to rescale by meters. The new 1 would then mean one meter, what we used to call about 3.1. It's pretty clear that the relation between the two scales is linear: For any point $\mathbf{v}$ on the line, we have the coordinates related by

$$3.1[\mathbf{v}]_m = [\mathbf{v}]_{ft}.$$

This is an example of change of basis. We found it easiest to remember the matrix S that has as its columns the "old" coords of the "new" basis vectors. Here the 1×1 matrix is just $[3.1]$. It follows that

$$3.1[\mathbf{x}]_m = [\mathbf{x}]_{ft}$$

for all $\mathbf{x}$, and $[\mathbf{x}]_m = S^{-1}[\mathbf{x}]_{ft}$. In $\mathbb{R}^1$, a change of basis is just a rescaling. The 1×1 matrix of a linear transformation of $\mathbb{R}^1$ is independent of basis.

2. A basis for $\mathbb{R}^2$ consists of having a pair of ordered marked axes. However, a change of coordinates (i.e., of bases) can be much more than rescaling the two axes. You can move them around, doing anything you want **except making the two axes coincide.**

We called S the matrix whose columns are the original (standard) coordinates of the vectors in the new ordered basis—hmm, my mind has forgotten that $n = 2$, but it doesn't seem to matter at all. Anyway, if the new basis is $\mathcal{B} = \{\vec{v}_1, \vec{v}_2\}$,

$$S = [\, \vec{v}_1 \mid \vec{v}_2 \,].$$

Conceptually, S is **the** matrix that takes $\vec{e}_i$ to $\vec{v}_i$ for $i = 1, 2$; the columns of a matrix ..., again: what a matrix does is completely determined by what it does to a basis of the domain. You may have to work awhile to *calculate* the other values the linear transformation, but it's routine stuff. And S^{-1} is **the** matrix that takes $\vec{v}_i$ to $\vec{e}_i$, so it gives the coordinate mapping for $\mathcal{B}$.

How does that fit in with #1? It is

$$[\mathbf{x}]_{\mathcal{B}} = S^{-1}[\mathbf{x}]_{st}.$$

To justify this, we could rewrite it as $[\mathbf{x}]_{st} = S[\mathbf{x}]_{\mathcal{B}}$. Then one sees that both sides agree on the two vectors of $\mathcal{B}$, which is enough (why?).

It goes the same way in every $\mathbb{R}^n$. The $n \times n$ matrix of a linear transformation of $\mathbb{R}^n$ typically depends on basis ($B = S^{-1}AS$) when $n > 1$.

Coordinates from Ch. 4

As I said, there's virtually no difference in matters of coordinates in general linear spaces V and in $\mathbb{R}^n$. From lecture: the calculation that $\{1+x, x+x^2, x^2\}$ is a basis of P_2 is *exactly* the same as the one that shows that $\{\vec{e}_1+\vec{e}_2, \vec{e}_2+\vec{e}_3, \vec{e}_3\}$ is a basis of $\mathbb{R}^3$. We must solve the same system of linear equations for the coeffients in linear combinations in either case. Indeed, if we had any 3-dimensional linear space with basis $\mathfrak{A} = \{f_1, f_2, f_3\}$, the determination of whether $\{f_1+f_2, f_2+f_3, f_3\}$ is another basis comes down to same 3×3 system.

Coordinate mappings defined on an n-dimensional linear space V are *isomorphisms* (invertible linear transformations) $[\ ]_{\mathfrak{B}} : V \to \mathbb{R}^n$, a way of matching the elements of basis $\mathfrak{B} = \{f_1, ..., f_n\}$ of V with the standard basis of $\mathbb{R}^n$. Every linear combination of basis elements in V gets matched up with the corresponding linear combination of standard basis vectors $\vec{e}_i$. It is hard to tell V and $\mathbb{R}^n$ apart algebraically, since we can compute in coodinates. All one has to do in V is to replace the $\vec{e}_i$'s by the f_i's at the end.

Perhaps you can see now why it is so hard for me to tell apart $\mathbb{R}^6 = \mathbb{R}^{6\times 1}$, $\mathbb{R}^{2\times 3}$, $\mathbb{R}^{3\times 2}$, $\mathbb{R}^{1\times 6}$, P_5, etc. From the viewpoint of our course, they are all just linear spaces of dimension 6, requiring six coordinates to describe their elements.

Ample reason for changing coordinates is given in the document "A tale of two bases." Refer to that: there is a basis $\mathfrak{B} = \{\vec{v}_1, \vec{v}_2\}$ of $\mathbb{R}^2$ with $T_A(\vec{v}_1) = 5\vec{v}_1$ and $T_A(\vec{v}_2) = 3\vec{v}_2$. Thus $[T_A]_{\mathfrak{B}}$ is a diagonal matrix. We will do that sort of thing systematically later in the course.