The singular values of a matrix

As I've written before, one hopes to diagonalize a general $m \times m$ matrix A, for diagonalizing a matrix is often very useful. A *symmetric* matrix A **always** diagonalizes with respect to some *orthonormal* basis. Don't misread this: it is easy to see how A gets diagonalized by some more general bases as well. It may help to note that a diagonal matrix is symmetric.

A *quadratic form* in $\mathbb{R}^m$ is the degree 2 analogue of a linear expression $\vec{a} \cdot \vec{x}$, where $\vec{a}$ is a constant vector. It can be written in the form

$$q_B(\vec{x}) = B\vec{x} \cdot \vec{x} = \vec{x} \cdot B\vec{x}$$

for some $m \times m$ matrix B, which we can and will take to be symmetric. The "definiteness" of B (or q_B) is determined by the signs of the eigenvalues of B.

When given an arbitrary $n \times m$ (so not necessarily square) matrix A, we can use the symmetric $m \times m$ matrix $B = A^T A$ to define a quadratic form as above, which can be rewritten as

$$(*) \qquad q_B(\vec{x}) = (A\vec{x}) \cdot (A\vec{x}) = \|A\vec{x}\|^2.$$

(When A is symmetric, then $B = A^2$.) Note that (*) gives at once that $q_B(\vec{x})$ is positive-semidefinite, and that $q_B(\vec{x}) = 0$ if and only if $\vec{x} \in \ker A$. Thus, when $\lambda = 0$ is an eigenvalue of B, its multiplicity equals the nullity of A. All other eigenvalues are positive. B can be diagonalized by an orthonormal basis $\{\vec{u}_1, ..., \vec{u}_m\}$, such that the corresponding eigenvalues of B in are listed in non-increasing order: $\lambda_1, ..., \lambda_m$.

Then $B\vec{u}_i \cdot \vec{u}_j = \lambda_i \delta_{i,j}$, so the vector $A\vec{u}_i$ has length $\sigma_i = (\lambda_i)^{1/2}$, and these numbers σ_i are called the *singular values* of A. From this, one can conclude that if 0 is not a singular value, the image of the unit *sphere* in $\mathbb{R}^m$ is an *ellipsoid* in the subspace image $(A) \subseteq \mathbb{R}^n$; when 0 is a singular value, the image is the region bounded by an ellipsoid. The maximum norm of a vector on this ellipsoid is σ_1. At issue are the values of the expression $\sum \lambda_i c_i^2$ when $\sum c_i^2 = 1$. We have

$$\sum \lambda_i c_i^2 \leq \sum \lambda_1 c_i^2 = \lambda_1 \sum c_i^2 = \lambda_1.$$