The heart of inner products

Let V be a linear space. Though it is not necessary for the definition, assume that V is finite dimensional and put $\dim V = n$. An *inner product* on V assigns a number to each pair $f, g \in V$ a real number, denoted $\langle f, g\rangle$. An inner product is to behave like dot product in $\mathbb{R}^n$:

a) $\langle f, g\rangle = \langle g, f\rangle$ (symmetry)

b) $\langle cf + h, g\rangle = c\langle f, g\rangle + \langle h, g\rangle$ (linearity with respect to the first entry, so likewise for the second by symmetry)

c) $\langle f, f\rangle \geq 0$, and $\langle f, f\rangle = 0$ if and only if $f = 0$ (positivity)

The simple fact, $\langle f, 0\rangle = 0$ for all $f \in V$, follows from b). We can rephrase b) as: *The mapping "inner product with any fixed g" is a linear transformation from V to* $\mathbb{R}$. Condition c) is used to define a notion of *distance* between elements of V.

Let $\mathfrak{B} = \{f_1, ..., f_n\}$ be a basis of V. It seems natural enough, given our experience with linear algebra, to think that the $n(n+1)$ numbers $b_{i,j} = \langle f_i, f_j\rangle$ (for $1 \leq i \leq j \leq n$) are enough to determine the inner product of any two vectors in V. This is correct, as it follows from a) and b). However, the numbers cannot be arbitrarily specified. The issue is that they must obey c) as well. For that, we must have

i) $b_{i,i} > 0$ for all i,

ii) $b_{i,j}^2 \leq b_{i,i} b_{j,j}$ (Cauchy-Schwarz),

but even that is not enough. The reason is that condition c) on a basis does not imply it for all $f \in V$. Find an example of this fact with $\dim V = 2$ (we were close to one in the Halloween lecture).

Last time I wrote: *When I'm doing a long calculation, I almost never get it right the first time around. Sometimes I don't even want to! And I don't like trying to follow someone else's calculation.* Can you imagine what I really meant by that?