Why isomorphisms?

First a word on an old theme: a linear transformation $V \to W$ is determined by its values on a basis. This is a jazzed up version of the easier statement for $\mathbb{R}^m \to \mathbb{R}^n$: an $n \times m$ matrix is determined by its values on the standard basis of $\mathbb{R}^m$ (the domain). We saw this explicitly: the i-th column vector of an $n \times m$ matrix A is $A\vec{e}_i$ (the value of T_A at $\vec{e}_i$). If that isn't clear by now, ...

All we have done is to extend this idea to cover arbitrary linear spaces and arbitrary bases. Declare where any one basis of V is to get mapped, and the rest of the domain follows the instructions given by linearity. There are other ways of specifying a function than giving a formula for it. Here, you are asked to tolerate the following small amount of abstraction: *we are specifying how the values of $T : V \to W$ get decided, without doing the tedious calculations that actually give the explicit outcome.*

Said another way, suppose you wanted to tell me about an exciting mapping from $\mathbb{R}^2$ to $\mathbb{R}^{35}$. Make it a linear mapping; call it T. Well, I need to hear something about T, for all I know otherwise is that $T(\begin{bmatrix} 0 \\ 0 \end{bmatrix}) = \vec{0} \in \mathbb{R}^{35}$. Then you say, trying to be helpful, "I'll give you a table of values:"

$$T(\begin{bmatrix} 1 \\ 0 \end{bmatrix}) = v_1 \quad \text{(something)},$$

continue $T(\begin{bmatrix} 1 \\ 1 \end{bmatrix}) = v_2$, $T(\begin{bmatrix} 1 \\ 2 \end{bmatrix}) = v_3$, ... At some point, I stop tuning out your flood of data, and exclaim, "Look, you've already mentioned a basis of $\mathbb{R}^2$ (take any two of the vectors in $\mathbb{R}^2$ given above) and gave me the corresponding values of T. I know what "linear" means, and I can take it from there to compute any other value of T (e.g., you must have $v_3 = -v_1 + 2v_2$ above) or even a formula for T—if and when I need to."

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The word *isomorphism* means a linear transformation that is both one-to-one and onto, hence has an inverse; the inverse is automatically linear. In how many useful equivalent ways can that be reworded? (Don't waste you time counting.) Here's a couple of equivalents to "$T : V \to W$ is an isomorphism":

1. The linear transformation $T : V \to W$ is onto and has $\ker T = 0$
2. The linear transformation $T : V \to W$ is one-to-one and has $\text{image}(T) = W$

We say that V and W are *isomorphic* if there is an isomorphism from one to the other. "Vague," you think? Well if there is an isomorphism going one direction, the inverse is an isomorphism mapping in the other direction. As with bases of a linear space, once there is one isomorphism in a non-silly situation ($V = \{0\}$ is the silly case), there are infinitely many; you can multiply one by any non-zero scalar!

In the case of finite dimensional spaces (which are the primary concern in this course), *V and W are isomorphic if and only if* $\dim V = \dim W$. (Use #1 or #2 above to see it as a consequence of rank-nullity.) It's really just a matter of pairing off a basis of V with a basis of W, and the rest of V gets assigned elements of W, following the instructions given by linearity. If we let n denote the common dimension of V and W, then the set of all isomorphisms is in rather natural correspondence with isomorphisms of $\mathbb{R}^n$ with itself, and that means the invertible $n \times n$ matrices. If it's so "natural", I should be able to show it to you—it's a deal, provided you accept the premises of linear algebra thus far.

Choose any bases $\mathfrak{B}$ of V and $\mathfrak{C}$ of W. It's a diagram again:

$$\begin{array}{ccc} V & \longrightarrow & W \\ {\scriptstyle [\ ]_{\mathfrak{B}}}\big\downarrow & & \big\downarrow{\scriptstyle [\ ]_{\mathfrak{C}}} \\ \mathbb{R}^n & \longrightarrow & \mathbb{R}^n \end{array}$$

Put into terms previously used in this course, an $n \times n$ matrix A gives the isomorphism $S : V \to W$,

$$S = [\ ]_{\mathfrak{C}}^{-1} T_A [\ ]_{\mathfrak{B}};$$

this transformation depends, of course, on the choice of $\mathfrak{B}$ and $\mathfrak{C}$. Conversely, an isomorphism $S : V \to W$ gives a matrix A, determined by the following isomorphism of $\mathbb{R}^n$ to itself:

$$[\ ]_{\mathfrak{C}} S [\ ]_{\mathfrak{B}}^{-1} = T_A.$$

(Here, I'm appealing to the fact that the composition of isomorphisms is an isomorphism.)

For plain finite sets, what is the analogous notion? It's "isomorphism" without the "linear": a one-to-one correspondence. Two finite sets can be put in one-to-one correspondence if and only if they have the same number of elements; when there are n elements in both sets, there are $n!$ ways of pairing them off (by a one-to-one correspondence). Non-trivial linear spaces have infinitely many elements, but also have the operations of linear algebra. We then concern ourselves with a basis, rather than the whole set. A linear space V can be "reconstituted" from any basis of V: just take all linear combinations of the basis elements.

You should view isomorphic linear spaces as being at bottom the same from the point of view of the two operations of linear spaces (vector addition and scalar multiplication). I mean, what's there to a (say) 6-dimensional linear space? find 6 linearly independent elements (a basis). Just pick one such set; which one is not so important (i.e., be a good gorilla and lay down some axes.) This gives you coordinates to work with. Again, make all linear combinations of the basis elements. You can say that *in the same manner* for all 6-dimensional linear spaces; you can make the deeds match for any two 6-dimensional linear spaces by first pairing off the sets of 6 elements, and matching (say) the sum of the first two elements of one to the sum of the first two elements of the other, and pairing linear combinations of 6 elements in the two spaces according to coefficients. It's the same as in $\mathbb{R}^6$, as far as the basic operations of linear algebra go. Thus, an isomorphism $V \to \mathbb{R}^6$ maps each linearly independent set in V to a linearly independent set in $\mathbb{R}^6$.

Coordinate transformations are **always** isomorphisms by their very definition. Recall that for V of dimension n and $\mathfrak{B} = \{v_1, ..., v_n\}$ a basis of V, $[\ ]_{\mathfrak{B}}$ is the one and only linear transformation $T : V \to \mathbb{R}^n$ with $T(v_i) = \vec{e_i}$ for $i = 1, ..., n$. Its inverse T^{-1} is the one and only linear transformation $\mathbb{R}^n \to V$ with $\vec{e_i} \mapsto v_i$ for $i = 1, ..., n$. We are pairing off basis vectors, so $T : V \to \mathbb{R}^n$ here **is** an isomorphism.

Maybe you can now see that *every* isomorphism $T : V \to \mathbb{R}^n$ is a coordinate mapping. Under the inverse of T, the standard basis vectors of $\mathbb{R}^n$ go *somewhere*; write this as $\vec{e_i} \mapsto v_i \in V$, and then $\mathfrak{B} = \{v_1, ..., v_n\}$ is necessarily a basis for V (why?) such that $T = [\ ]_{\mathfrak{B}}$.