ETHICS PLEDGE: I agree to complete this exam without unauthorized assistance from any person or person's work, materials or device.

Your name (print): ________________________ Section:________

Signature: ________________________________ Date: ________

1[30]	2[20]	3[20]	4[20]	5[20]	6[20]	7[20]	8[20]	9[10]	10[10]	11[10]	T [200]

110.201 Prof. Zucker FinalExam: Dec. 15, 2006 Time: 3 hours

Name: **Section Number:**

No books, no notes, no calculators or other electronic devices. Write legibly, and show all relevant work—or risk losing credit. Answer what is asked, and only what is asked.

[30] 1. Let $A = \begin{bmatrix} 2 & 3 & 2 \\ 1 & 2 & 2 \\ 0 & -1 & -2 \end{bmatrix}$.

a) Give the definition of "$\{\vec{v}_1, \vec{v}_2, ..., \vec{v}_k\}$ is a *linearly independent* set" **in terms of linear combinations or linear relations.** Use <u>this</u> definition to show that $\mathfrak{B} = \{\vec{e}_3, \vec{e}_1 + \vec{e}_2, \vec{e}_1 - \vec{e}_2\}$ is a *basis* of $\mathbb{R}^3$.

b) Determine the $\mathfrak{B}$-matrix of A, where $\mathfrak{B}$ is the basis in part a).

c) Determine all real numbers c for which the equation (for $\vec{x} \in \mathbb{R}^3$) $A\vec{x} = c\vec{x}$ has a non-trivial solution.

[20] 2. Let $A = \begin{bmatrix} 2 & 3 & 2 \\ 1 & 2 & 2 \\ 0 & 1 & 2 \\ 0 & -1 & -2 \end{bmatrix}$.

a) Determine all vectors $\vec{x}$ that satisfy the linear system $A\vec{x} = \vec{0}$.

b) Explain why $W = \{\vec{b} \in \mathbb{R}^4 : A\vec{x} = \vec{b} \text{ is consistent}\}$ is a *subspace* of $\mathbb{R}^4$.

c) Let W be as in part b). Show that $\vec{b}_1 = 3\vec{e}_1 + 2\vec{e}_2 + \vec{e}_3 - \vec{e}_4 \in W$.

d) Let $\vec{b}_1$ be as in part c). Let $\vec{x}_1$ be **one** vector that satisfies $A\vec{x}_1 = \vec{b}_1$. Explain why the solution set of $A\vec{x} = \vec{b}_1$ equals the set of all vectors of the following form: $\vec{x}_1$ plus a solution of the equation in part a).

[20] 3. a) Let $A = \begin{bmatrix} 4 & -3 \\ -3 & -4 \end{bmatrix}$. Determine a diagonal matrix D with real entries, and an orthogonal matrix S for which $A = SDS^{-1}$. **Multiply out** SDS^{-1} to check that your answer is correct.

b) Let $B = \begin{bmatrix} 4 & 3 \\ -3 & -4 \end{bmatrix}$. Determine whether B is similar to the matrix A (from part a) in $\mathbb{R}^{2\times 2}$.

[20] 4. Let P_4 be, as usual, the linear space of polynomials of degree ≤ 4.

a) Specify an isomorphism $\Phi : P_4 \to \mathbb{R}^n$ for some n. Explain why it is an isomorphism.

b) *True or false*: Every linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ is an isomorphism. **Explain.**

c) *True or false*: There exists an inner product on P_4 with the property that $\{1, t, t^2, t^3, t^4\}$ is an orthonormal set. **Explain.**

d) (Convince yourself that $\mathfrak{A} = \{2,\ 3 - t,\ 4t - t^2,\ 5 - t^3,\ 6 - t^4\}$ is a basis of P_4.) Determine the $\mathfrak{A}$-coordinate vector of $t^4 + t$.

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[20] 5. a) Let V be a linear space. Suppose that λ is an eigenvalue of the linear transformation $T : V \to V$. Derive the fact that λ^2 is an eigenvalue of T^2.

b) Determine all matrices in $\mathbb{R}^{3\times 3}$ that are both symmetric and orthogonal, and describe them geometrically. [*Suggestion:* Express the two conditions in terms of "transpose."]

[20] 6. Determine the matrix (for the standard basis of $\mathbb{R}^5$) of the orthogonal projection of $\mathbb{R}^5$ onto the "plane" with equations $x_1 - x_5 = 0,\ \ x_1 + x_2 + x_3 + x_4 = 0$.

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[20] 7. For which $a \in \mathbb{R}$, $b \in \mathbb{R}$ does the matrix $A = \begin{bmatrix} 2 & 0 & 0 \\ b & 1 & 0 \\ 0 & a & 1 \end{bmatrix}$ have an eigenbasis (for $\mathbb{R}^3$)? *When it does, specify an eigenbasis (depending on a and b).*

[5,5,10] 8. Let V be $Span\{1, \sin x, \cos x\}$. The dimension of V is 3. We define a linear transformation $T : V \to \mathbb{R}^4$ by $T(f) = 2f(0)\vec{e}_1 + f(\pi)\vec{e}_2 - f(2\pi)\vec{e}_3$.

a) Determine the rank of T.

b) Determine a basis for the kernel of T.

c) Let D denote the linear operator on V given by $D(f) = f'$. Determine the complex eigenvalues of D—that includes the real ones!—and the corresponding eigenspaces.

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[10] 9. Let $A = \begin{bmatrix} 1 & b \\ c & 1 \end{bmatrix}$, where b and c are real scalars. Determine the set of values of b and c for which the dynamical system $\vec{x}(t+1) = A\vec{x}(t)$ is asymptotically stable (meaning: for all initial states, the state vector tends to $\vec{0}$ as $t \to \infty$).

[10] 10. Determine whether $x_1^2 + 3x_1x_2 + 2x_2^2 = 1$ is the equation of an ellipse.

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[10] 11. a) Give an example of two 2×2 real matrices that have the same characteristic polynomial yet they are not similar. **Explain.**

b) *True or False:* If a matrix fails to diagonalize over $\mathbb{R}$, it will diagonalize over $\mathbb{C}$. **Explain.**