A few comments on inner product spaces

Let V be a linear space of dimension n, as suppose an inner product $< f, g >$ on V is given. How might one picture that? Like $\mathbb{R}^n$ with dot product; maybe picture the latter by $\mathbb{R}^3$ with dot product. But also know the limitations of these pictures.

Recall that an inner product is something that obeys the same rules that dot product does:

i) $<g, f> = <f, g>$,

ii)$< f, g >$ is bilinear in f and g;

iii)$<f, f> \geq 0$, and equals 0 if and only if $f = \vec{0}$.

The third one is different from the other two, involving *inequalities.* I want to outline some of the consequences of i) and ii), and then turn to iii):

1. Let $\mathfrak{B} = \{f_1, ..., f_n\}$ be a basis of V. Then the inner product is completely determined by the numbers $a_{ij} =< f_i, f_j >$. This one might call *bilinearity*. By i), $a_{ij} = a_{ji}$. The formula: if A is the symmetric $n \times n$ matrix with entries a_{ij}, and $[\ ]_{\mathfrak{B}}$ is the coordinate mapping $V \to \mathbb{R}^n$, then $< f, g >= [f]_{\mathfrak{B}}^T A[g]_{\mathfrak{B}}$. Conversely, the coordinate mapping, which is given the same way, inner product or not, gives a way to use dot product on $\mathbb{R}^n$ to make an inner product on V. I did that in lecture.

2. V possesses orthonormal bases, for which $A = I$ $(a_{ij} = \delta_{ij})$. You can get one from any basis by the Gram-Schmidt process. In that case, the "identity theft" by V via $[\ ]_{\mathfrak{B}}$ also "steals" (perhaps "copies" is more accurate) dot product from $\mathbb{R}^n$. Note that dot product can be written in terms of matrix multiplication as $x \cdot y = x^T I y$.

Next, iii) is quite sneaky. We know that Cauchy-Schwarz, a *very important* inequality,[1] gives

$$|<f, g>| \leq \|f\| \, \|g\| \tag{1}$$

when f and g are in V. Suppose we have a basis of unit vectors for the inner product, i.e., $a_{ii} = 1$ for all $1 \leq i \leq n$. Then (1) implies that $|a_{ij}| \leq 1$ for all i, j. In the simple case of an inner product on $\mathbb{R}^2$, let's see directly why $a_{12} = 2$ is ruled out. For instance,

$$< (e_1 - e_2), (e_1 - e_2) > = <e_1, e_1> + <e_2, e_2> -2 <e_1, e_2>$$

would then equal $-2 < 0$, violating iii). Check that we can't even have $a_{12} = 1$.

[1] Inequalities provide a way to assure correctness when exact quantities are unavailable or unnecessary.