A tale of two bases

Let's start with a tale of *one* basis of some ... why confound things? a basis of $\mathbb{R}^n$. Get down to the bottom of things: let $n = 2$. I hope you see that $n = 1$ is too easy for most things, **but not now!** So let $n = 1$.

If I told you that we had a Calc 0 function $f : \mathbb{R}^1 \to \mathbb{R}$, and that $f(1) = 2$; determine f. You'd think I was toying with you. You'd say, there are infinitely many such functions, e.g. $f(x) = 2x^k$ for every $k > 0$. And that's absolutely correct. But if I added that f is linear, the only answer would be $f(x) = 2x$.

The message you should have picked up by now is the notion of linearity, as I put it. That's Fact 1.3.9. It follows that *a linear mapping is completely determined by its value on any one basis.* Never forget that! Since $\{1\}$ is a basis of $\mathbb{R}^1$, meaning that every number is a multiple of 1, linearity gives $f(c) = cf(1)$; $f(1) = 2$ thereby determines $f(x) = 2x$.

Let's move onward to $\mathbb{R}^2$ function of two variables. Suppose I have a secret function $f : \mathbb{R}^2 \to \mathbb{R}$ and I told you (say) $f(1,1) = 2$ and $f(1,-2) = 3$. Tell me my function or you die! That would reduce attendence below that in recent lectures. But if I told you the miraculous feature that f is linear, there is sufficient information; there is only one such linear f. (Which one?) Why? Because $\mathcal{B} = \{(\vec{e}_1 + \vec{e}_2), (\vec{e}_1 - 2\vec{e}_2)\}$ is a *basis* for $\mathbb{R}^2$. Linearity tells you how to deduce the value of f at every vector of $\mathbb{R}^2$.

The above contains the essential issues of Linear Algebra, and we have not gotten beyond $\mathbb{R}^2$ as domain and $\mathbb{R}^1$ as codomain!

Here comes the tale of two bases.

– Z: Which vector in $\mathbb{R}^2$ has coordinates 1,0?

– A: [$\vec{e}_1$ of course.]

Don't you have to specify a basis?

– Z: Yes, of course. The basis I have in mind is $\mathcal{B} = \{(\vec{e}_1 + \vec{e}_2), (\vec{e}_1 - 2\vec{e}_2)\}$.

– A: Why that one?

– Z: [Do you need a reason?]

It will become clearer later on. I have something up my sleeve.

– A: OK. The answer is the first element of your [stupid] basis, namely $\vec{e}_1 + \vec{e}_2$.

– Z: Good, now you're getting serious. Next, tell me the matrix S of the coordinate mapping with respect to the basis $\mathcal{B}$.

– A: OK. Isn't that the one where you put the vectors of $\mathcal{B}$ in order as the columns of S:

(*)
$$S = \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix}$$

[*#*!] Or is that S^{-1}? ....

– Z: Stay calm. The matrix you gave takes $\vec{e}_1$ to $\vec{e}_1+\vec{e}_2$, and $\vec{e}_2$ to $\vec{e}_1-2\vec{e}_2$. That's backwards, right? The coordinate map takes any vector in $\mathbb{R}^2$ to the coefficients of the basis under consideration. You really want $\vec{e}_1+\vec{e}_2$ to map to $\vec{e}_1$, etc. So you have written down the inverse matrix, S^{-1}. Can you invert a 2×2 matrix?

– A: Do you take me for a dummy?

– Z. [No comment.] Of course not. Do you see the point of linearity? If you have a linear transformation of $\mathbb{R}^m$, that is completely specified by its value on m linearly independent elements. Just take linear combos and invoke linearity. (Conceptually, you can think that without carrying this out in numbers. Do you really want to carry that out in (say) $\mathbb{R}^{10}$?)

– A: I think I'm getting the point.

– Z. Great! Now, let's determine the matrix of T_A, where A is the 2×2 matrix:

$$A = \tfrac{1}{3} \begin{bmatrix} 13 & 2 \\ 4 & 11 \end{bmatrix}$$

– A: It's A, of course.

– Z: That's for the standard basis. I've been insisting on my basis $\mathcal{B}$. You'll see why.

We abridge the ending. The matrix is

$$B = \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix}$$

As the theory produces unambiguously, $B = SAS^{-1}$. Check it out. The punch line: that mess of a matrix A is describing a transformation that stretches by a factor of 5 in one direction (of $\vec{v}_1 = \vec{e}_1 + \vec{e}_2$), by a factor of 3 in another ($\vec{v}_2 = \vec{e}_1 - 2\vec{e}_2$). Isn't that a clearer picture of what T_A does? (So what if these directions aren't perpendicular.)

There is a procedure for making transformations look simpler, and it comes later in the course.