1. *(15 points)* Find the trigonometric polynomial $p(t) = a + b\sin t + c\cos t$ of degree 1 which best fits the data:

t	$y(t)$
0	7
$\pi/2$	0
π	0
$3\pi/2$	0
2π	0

2. *(15 points)* For the hyperbola

$$7x_1^2 - 6x_1x_2 - x_2^2 = 8$$

find:

(a) the principal axes,

(b) the equation of the hyperbola in the coordinate system given by the principal axes, and

(c) the asymptotes. [*Hint.* The asymptotes of a hyperbola $q(\vec{x}) = 1$ are the diagonals of the rectangle whose vertices are

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} \frac{\pm 1}{\sqrt{|\lambda_1|}} \\ \frac{\pm 1}{\sqrt{|\lambda_2|}} \end{bmatrix}$$

in c_1-c_2 coordinates (principal axes coordinates).]

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3. *(15 points)* Consider the matrix

$$A = \begin{bmatrix} 0 & 0 & 0 \\ -2 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix}$$

(a) Find all the (real or complex) eigenvalues of A.

(b) Diagonalize the matrix A (over the complex numbers, if necessary).

4. *(20 points)* Consider the space $\mathbb{R}^{2\times 2}$ of 2×2 matrices. Recall that the standard basis of $\mathbb{R}^{2\times 2}$ is given by $\mathfrak{E} = \{E_{11}, E_{12}, E_{21}, E_{22}\}$.

   Consider also the linear transformation $L : \mathbb{R}^{2\times 2} \to \mathbb{R}^{2\times 2}$ given by

   $$L(A) = \frac{1}{2}(A + A^T).$$

   (a) Find the matrix $M = [L]_{\mathfrak{E}}$ (M is symmetric!).

   (b) Assume as a fact that the linear transformation L is the orthogonal projection[1] onto some subspace $\mathcal{S} \subset \mathbb{R}^{2\times 2}$. Find a basis $\mathfrak{B}$ for $\mathcal{S}$. [*Hint.* L must be the orthogonal projection onto its own image, so start by finding a basis for $\text{Im}(M)$.]

   (c) Find an *orthonormal* basis $\mathfrak{U}$ for $\mathcal{S}^\perp$ (the orthogonal complement of the subspace $\mathcal{S}$ with respect to the inner product in $\mathbb{R}^{2\times 2}$). [*Hint.* Since L is an orthogonal projection, $\mathcal{S}^\perp = \text{Ker}(L)$, so start by finding a basis for $\text{Ker}(M)$.]

   (d) Write down a formula for the orthogonal projection $P : \mathbb{R}^{2\times 2} \to \mathbb{R}^{2\times 2}$ onto $\mathcal{S}^\perp$. [*Hint.* The orthonormal basis $\mathfrak{U}$ of (c) might help.]

[1] With respect to the inner product $\langle A, B\rangle = \text{Trace}(A^T B)$ in $\mathbb{R}^{2\times 2}$.

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5. **TRUE OR FALSE.** *(5 points each)* Justify your answers!

(a) If all the (real or complex) eigenvalues of A are zero, then A is the zero matrix.

(b) If A is a (square) skew-symmetric matrix, then $\det(A) = 0$.

(c) If A, B are symmetric matrices, so is their product AB.

(d) If $A_{n \times n}$ is diagonalizable (over the real numbers) then A is similar to a symmetric matrix.

(e) If $A^2 = 0$ for a 10×10 matrix A, then the inequality $\text{rank}(A) \leq 5$ must hold. [*Hint.* Justify the following first: for such an A we have $\text{Im}(A) \subset \text{Ker}(A)$. After that, the rank-nullity theorem may help.]

(f) If A is a symmetric matrix and $\vec{x} \in \text{Ker}(A), \vec{y} \in \text{Im}(A)$ then $\vec{x} \perp \vec{y}$. [*Hint.* Use the fundamental theorem of linear algebra, or else do a direct calculation.]

(g) There is a *symmetric* matrix A such that $A \neq 0$ and $A^2 = 0$. [*Hint.* How does A being symmetric help?]