1. *(15 points)* Find the quadratic polynomial $p(t) = a + bt + ct^2$ which best fits the data:

t	$y(t)$
-2	-4
-1	-1
0	0
1	0
2	0

2. *(15 points)* The matrix

$$A = \frac{1}{9}\begin{bmatrix} 8 & 2 & -2 \\ 2 & 5 & 4 \\ -2 & 4 & 5 \end{bmatrix}$$

is the matrix of the orthogonal projection onto some subspace $V \subset \mathbb{R}^3$.

(a) Find an orthonormal basis for V.

(b) Find an orthonormal basis for $V^\perp$.

(c) Find the matrix P of the orthogonal projection onto $V^\perp$.

3. *(15 points)* For the ellipse

$$6x_1^2 + 4x_1 x_2 + 3x_2^2 = 1$$

find:

(a) the principal axes,

(b) the equation of the ellipse in the coordinate system given by the principal axes, and

(c) the lengths of the semiaxes.

4. *(20 points)* Consider the following quadratic form in $\mathbb{R}^2$:

$$q(\vec{x}) = q(x_1, x_2) = \vec{x}^T A \vec{x} = 2x_1^2 - 4x_1x_2 + 5x_2^2, \qquad A = \begin{bmatrix} 2 & -2 \\ -2 & 5 \end{bmatrix}.$$

Define also

$$\langle \vec{v}, \vec{w} \rangle = \vec{v}^T A \vec{w} = 2v_1w_1 - 2v_1w_2 - 2v_2w_1 + 5v_2w_2. \qquad (1)$$

Observe that $q(\vec{x}) = \langle \vec{x}, \vec{x} \rangle$.

(a) Suppose we set out to prove that $\langle \vec{v}, \vec{w} \rangle$ is an inner product in $\mathbb{R}^2$. Assume as known that, for any vectors $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^2$ and scalar c,

- $\langle \vec{v}, \vec{w} \rangle = \langle \vec{w}, \vec{v} \rangle$,
- $\langle \vec{u} + \vec{v}, \vec{w} \rangle = \langle \vec{u}, \vec{w} \rangle + \langle \vec{v}, \vec{w} \rangle$, and
- $\langle c\vec{v}, \vec{w} \rangle = c\langle \vec{v}, \vec{w} \rangle$.

What else needs to be shown in order to complete the proof that $\langle \vec{v}, \vec{w} \rangle$ is an inner product? State it and prove it. [*Hint:* you will have to determine what type of quadratic form q is.]

(b) By (a), $\mathbb{R}^2$ has an inner product given by (1). Determine whether or not the standard basis $\mathfrak{E} = \{\vec{e}_1, \vec{e}_2\}$ is an orthonormal basis of $\mathbb{R}^2$ *with respect to the inner product* (1) and, if not, find an orthonormal basis $\mathfrak{U} = \{\vec{u}_1, \vec{u}_2\}$.

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5. **TRUE OR FALSE.** *(5 points each)* Justify your answers!

(a) If A and B are 2×2 matrices then the eigenvalues of AB and BA are the same. [*Hint.* Compare the characteristic polynomials of AB and BA.]

(b) $A = \begin{bmatrix} 3/5 & -4/5 \\ -4/5 & -3/5 \end{bmatrix}$ is the matrix of a reflection. [*Hint.* How can we decide if A is a reflection using its eigenvalues and eigenvectors?]

(c) If $T : \mathbb{R}^5 \to \mathbb{R}^5$ is a linear transformation then there is a basis $\mathfrak{B}$ of $\mathbb{R}^5$ such that $[T]_{\mathfrak{B}} = I_5$.

(d) If $V \subset \mathbb{R}^n$ is an arbitrary subspace then there exists a matrix A such that $\mathrm{Im}(A) = V$.

(e) If A, B are $n \times n$ (symmetric) positive definite matrices, then $A + B$ is also positive definite. [*Hint.* Do *not* reason in terms of eigenvalues.]

(f) If $A^T\vec{b} = \vec{0}$ then system $A\vec{x} = \vec{b}$ is consistent.

(g) If $A_{2\times 2}$ is the matrix of a shear then $A^2 + I_2 = 2A$. [*Hint.* Choose an appropriate change of basis taking A into a simpler matrix B and reason in terms of B first, then revert back to the matrix A.]