Inverting 2×2 matrices

In this note we invert the general 2×2 matrix as in Theorem 1.4.5 of Anton–Rorres. However, we apply only the standard inversion method, with no guesswork or ingenuity needed.

THEOREM 1 *The 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is invertible if and only if $\Delta \neq 0$, where we write $\Delta = ad - bc$. When $\Delta \neq 0$, the inverse is*

$$A^{-1} = \frac{1}{\Delta}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Proof We row reduce the 2×4 partitioned matrix

$$[A|I] = \left[\begin{array}{cc|cc} a & b & 1 & 0 \\ c & d & 0 & 1 \end{array}\right] \tag{2}$$

to obtain the reduced row echelon matrix $[I|A^{-1}]$. There are two cases, depending on whether $a = 0$ or not.

Case $a \neq 0$ We multiply row 1 by $1/a$ to get

$$\left[\begin{array}{cc|cc} 1 & b/a & 1/a & 0 \\ c & d & 0 & 1 \end{array}\right]$$

Then we subtract c times row 1 from row 2 to obtain

$$\left[\begin{array}{cc|cc} 1 & b/a & 1/a & 0 \\ 0 & \Delta/a & -c/a & 1 \end{array}\right]$$

where we write the entry at row 2, column 2 as $d - bc/a = \Delta/a$. If $\Delta = 0$, inversion breaks down at this point, as we will not get a leading 1 in column 2; otherwise, we multiply row 2 by a/Δ to get the row echelon form

$$\left[\begin{array}{cc|cc} 1 & b/a & 1/a & 0 \\ 0 & 1 & -c/\Delta & a/\Delta \end{array}\right]$$

Finally, we subtract b/a times row 2 from row 1 to get the desired *reduced* row echelon matrix, whose right half we read off as A^{-1}, in the required form,

$$\left[\begin{array}{cc|cc} 1 & 0 & d/\Delta & -b/\Delta \\ 0 & 1 & -c/\Delta & a/\Delta \end{array}\right]$$

where at row 1, column 3 we write

$$\frac{1}{a} + \frac{bc}{a\Delta} = \frac{ad - bc + bc}{a\Delta} = \frac{d}{\Delta}$$

Case $a = 0$ We must have $c \neq 0$ for inversion to progress, otherwise we have a column of zeros and will never get a leading 1 in column 1. First we switch rows 1 and 2 in (2),

$$\left[\begin{array}{cc|cc} c & d & 0 & 1 \\ 0 & b & 1 & 0 \end{array}\right]$$

Now we multiply row 1 by $1/c$ to get the leading 1 in row 1,

$$\left[\begin{array}{cc|cc} 1 & d/c & 0 & 1/c \\ 0 & b & 1 & 0 \end{array}\right]$$

Again, inversion breaks down here unless $b \neq 0$ because we need a leading 1 in column 2. We multiply row 2 by $1/b$ to get the row echelon matrix

$$\left[\begin{array}{cc|cc} 1 & d/c & 0 & 1/c \\ 0 & 1 & 1/b & 0 \end{array}\right]$$

Finally, we subtract d/c times row 2 from row 1 to obtain the reduced row echelon matrix

$$\left[\begin{array}{cc|cc} 1 & 0 & -d/bc & 1/c \\ 0 & 1 & 1/b & 0 \end{array}\right]$$

Because now $\Delta = -bc$, this is what we want. (The condition $\Delta \neq 0$ is exactly what we need to guarantee that $c \neq 0$ and $b \neq 0$.) □

Because the expression Δ occurs everywhere, it deserves a name.

DEFINITION 3 The *determinant* $\det(A)$ of the 2×2 matrix A is the expression

$$\det(A) = \Delta = ad - bc$$

The method generalizes in principle to produce a formula for the inverse of a general $n\times n$ matrix, so we know a formula exists. Even for $n = 3$, we need a better way to find it.