Inverse Linear Substitutions and Matrices

We generalize Example 3 (on page 5 of Anton–Rorres) by solving the linear system

$$\left\{\begin{array}{rrrcrrrl} x & +y & +2z & = & u & & & (1)\\ 2x & +4y & -3z & = & & v & & (2)\\ 3x & +6y & -5z & = & & & w & (3)\end{array}\right. \qquad \text{with matrix } A = \begin{bmatrix} 1 & 1 & 2\\ 2 & 4 & -3\\ 3 & 6 & -5\end{bmatrix}$$

for x, y, and z in terms of u, v, and w. In the book, $u = 9$, $v = 1$, and $w = 0$. The point is that we can solve the more general problem with very little extra work, by following exactly the same steps as before (annotated symbolically).

$$\begin{array}{rrrcrrrr|l}
 & 2y & -7z & = & -2u & +v & & (4) & = (2) - 2(1)\\
 & 3y & -11z & = & -3u & & +w & (5) & = (3) - 3(1)\\
 & y & -\frac{7}{2}z & = & -u & +\frac{1}{2}v & & (6) & = \frac{1}{2}(4)\\
 & & -\frac{1}{2}z & = & & -\frac{3}{2}v & +w & (7) & = (5) - 3(6)\\
 & & z & = & & 3v & -2w & (8) & = -2(7)\\
x & & +\frac{11}{2}z & = & 2u & -\frac{1}{2}v & & (9) & = (1) - (6)\\
x & & & = & 2u & -17v & +11w & (10) & = (9) - \frac{11}{2}(8)\\
 & y & & = & -u & +11v & -7w & (11) & = (6) + \frac{7}{2}(8)
\end{array}$$

Thus the desired solution consists of equations (10), (11), and (8),

$$\left\{\begin{array}{rcrrr} x & = & 2u & -17v & +11w\\ y & = & -u & +11v & -7w\\ z & = & & 3v & -2w\end{array}\right. \qquad \text{with matrix } B = \begin{bmatrix} 2 & -17 & 11\\ -1 & 11 & -7\\ 0 & 3 & -2\end{bmatrix}$$

This is the *inverse* linear substitution, whose matrix B is the *inverse matrix* A^{-1} to A. In matrix notation, we solved the equation $A\mathbf{x} = \mathbf{u}$ and obtained the solution $\mathbf{x} = B\mathbf{u}$, where we write

$$\mathbf{x} = \begin{bmatrix} x\\ y\\ z\end{bmatrix}, \quad \mathbf{u} = \begin{bmatrix} u\\ v\\ w\end{bmatrix}$$

Computationally, we in effect applied row reduction to the 3×6 partitioned matrix $[A|I]$ to produce the reduced row echelon matrix $[I|B]$,

$$\left[\begin{array}{rrr|rrr} 1 & 1 & 2 & 1 & 0 & 0\\ 2 & 4 & -3 & 0 & 1 & 0\\ 3 & 6 & -5 & 0 & 0 & 1\end{array}\right] \longrightarrow \left[\begin{array}{rrr|rrr} 1 & 0 & 0 & 2 & -17 & 11\\ 0 & 1 & 0 & -1 & 11 & -7\\ 0 & 0 & 1 & 0 & 3 & -2\end{array}\right]$$

(If it had failed to reduce to this form, the message would have been that A was not invertible.)