Final Examination

Alternate Edition

Three hours. Closed book. No notes. 200 points, 20 per question.

Partial credit may be available, but only if you show your working.

Begin each of the ten questions on a new page and number it clearly in the margin.

If you use two books, label them "Book 1 of 2" and "Book 2 of 2". (If you use three books, . . .)

Calculators are allowed but not needed or recommended. Do not give decimal approximations for square roots or trigonometric functions; leave them as is.

There are many opportunities for you to check your work.

1. (a) Find the lengths of the edges of the triangle in $\mathbf{R}^4$ with vertices $P = (1, 0, 1, 3)$, $Q = (3, 1, 1, 1)$, and $R = (2, 2, 3, 5)$.

(b) Show that PQ is orthogonal to PR.

(c) State Pythagoras's Theorem for this triangle, and *write down* the verification that it holds.

2. (a) Find an *orthogonal* matrix P such that $D = P^{-1}AP$ is a diagonal matrix, where

$$A = \begin{bmatrix} 4 & 5 \\ 5 & 4 \end{bmatrix}$$

(b) Write down the resulting matrix D (for your choice of P).

3. Solve the linear system

$$\begin{cases} x & -y & +2z & -2t & = & 4 \\ 2x & -2y & +z & +2t & = & 5 \\ x & -y & +z & & = & 3 \end{cases}$$

4. (a) Find a basis of the row space of the matrix

$$C = \begin{bmatrix} 1 & 2 & 3 & 1 & 2 & 3 \\ 4 & 5 & 6 & 4 & 5 & 6 \\ 3 & 2 & 1 & 3 & 2 & 1 \\ 6 & 5 & 4 & 6 & 5 & 4 \end{bmatrix}$$

(b) Find a basis of the column space of the matrix C.

OVER →

5. In each part, either write down *one* example of a set of vectors in $\mathbf{R}^4$ that satisfies the requested conditions, *or* give a short reason why no such set exists.

(a) A set S_2 of two nonzero vectors in $\mathbf{R}^4$ that is linearly independent.

(b) A set S_3 of three vectors in $\mathbf{R}^4$ that is not linearly independent.

(c) A set S_4 of four vectors that is linearly independent but does not span $\mathbf{R}^4$.

(d) A set S_5 of five vectors (with no repetitions) that spans $\mathbf{R}^4$.

(e) A set S_6 of six vectors in $\mathbf{R}^4$ that is linearly independent.

6. (a) Find the point L in the plane $2x + y - 2z = 1$ in $\mathbf{R}^3$ that is closest to the point $K = (1, 2, 1)$. (Hint: Do not use calculus.)

(b) Find the angle between OK and LK (where O denotes the origin).

7. Apply the Gram–Schmidt process to find an *orthonormal* basis of the subspace W of $\mathbf{R}^4$ spanned by the vectors $(1, 1, -1, 1)$ and $(1, -2, -1, 2)$. (Hint: Treat the vectors in this order, and the answer will not require any square roots.)

8. (a) Find the eigenvalues of the matrix

$$F = \begin{bmatrix} 4 & 0 & -1 \\ c & 2 & 3 \\ 2 & 0 & 1 \end{bmatrix}$$

where c is unknown.

(b) For what value(s) of c is the matrix F diagonalizable?

9. Denote by P_4 the vector space of all polynomials in x of degree 4 (or less), and consider the linear transformation $T\colon P_4 \to P_4$ given by

$$T(f(x)) = f'(x) + f''(x)$$

where $f(x)$ is any polynomial in P_4. (Yes, f' denotes the derivative of f.)

Hint: First calculate $T(a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0)$.

(a) Find a basis of the kernel of T.

(b) Find a basis of the range of T.

(c) What is the rank of T?

(d) What is the nullity of T?

10. By any method, find the inverse of the matrix

$$G = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 2 & 0 \\ 4 & 9 & 1 \end{bmatrix}$$