Diagonalization

References are to Anton–Rorres, 7th Edition

Let $T\colon \mathbf{R}^n \to \mathbf{R}^n$ be a linear transformation, where we treat the vectors in $\mathbf{R}^n$ as *column* vectors. By Theorem 4.3.3, there is a unique $n\times n$ matrix A such that T is expressed in terms of A by

$$\mathbf{y} = T(\mathbf{x}) = A\mathbf{x} \tag{1}$$

In the reverse direction, we write $A = [T]_E$, the *matrix of T with respect to the standard basis* $E = \{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$. Explicitly, A is the matrix

$$A = [T(\mathbf{e}_1)|T(\mathbf{e}_2)|\ldots|T(\mathbf{e}_n)] \tag{2}$$

whose j-th column is the image $T(\mathbf{e}_j)$ of $\mathbf{e}_j$ under T. Expanding this column, we have

$$T(\mathbf{e}_j) = a_{1j}\mathbf{e}_1 + a_{2j}\mathbf{e}_2 + \ldots + a_{nj}\mathbf{e}_n \tag{3}$$

Change of basis We wish to simplify the matrix by changing to a different basis $B = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ of $\mathbf{R}^n$. Any vector $\mathbf{x} \in \mathbf{R}^n$ can be expressed uniquely as a linear combination

$$\mathbf{x} = x'_1\mathbf{v}_1 + x'_2\mathbf{v}_2 + \ldots + x'_n\mathbf{v}_n \tag{4}$$

for suitable coefficients $x'_i \in \mathbf{R}$.

Definition 5 The *coordinate vector* $[\mathbf{x}]_B$ *of* $\mathbf{x}$ *with respect to the new basis* B is the vector

$$\mathbf{x}' = [\mathbf{x}]_B = \begin{bmatrix} x'_1 \\ x'_2 \\ \vdots \\ x'_n \end{bmatrix}$$

Definition 6 The *matrix* $[T]_B$ *of T with respect to the new basis B* is the $n\times n$ matrix D whose entries d_{ij} satisfy the analogue

$$T(\mathbf{v}_j) = d_{1j}\mathbf{v}_1 + d_{2j}\mathbf{v}_2 + \ldots + d_{nj}\mathbf{v}_n \tag{7}$$

of equation (3) for each j.

Let $\mathbf{y}' = [\mathbf{y}]_B$ be the new coordinate vector of $\mathbf{y} = T(\mathbf{x})$. Then the analogue of equation (1) is

$$\mathbf{y}' = D\mathbf{x}' \tag{8}$$

The transition matrix We need to know how to compute the new matrix D from A, and the new coordinate vector $\mathbf{x}'$ from $\mathbf{x}$; also vice versa. Obviously, we have to use the $\mathbf{v}$'s in some way, so we encode them in a matrix.

Definition 9 The *transition matrix from the basis B to the basis E* is the $n\times n$ matrix

$$P = [\mathbf{v}_1|\mathbf{v}_2|\ldots|\mathbf{v}_n]$$

whose j-th column is the vector $\mathbf{v}_j$.

Note the *direction*; P expresses the *new* basis B in terms of the *old* standard basis E, not the other way round. To relate $\mathbf{x}$ and $\mathbf{x}'$, we take the i-th coordinate in equation (4), noting that we have encoded the i-th coordinate of the vector $\mathbf{v}_j$ as the entry p_{ij} in P,

$$\begin{aligned} x_i &= x'_1 p_{i1} + x'_2 p_{i2} + \ldots + x'_n p_{in} \\ &= p_{i1} x'_1 + p_{i2} x'_2 + \ldots + p_{in} x'_n \end{aligned}$$

which we recognize as the entry in row i of the matrix product $P\mathbf{x}'$. Thus

$$\mathbf{x} = P\mathbf{x}' \tag{10}$$

We also need to express $\mathbf{x}'$ in terms of $\mathbf{x}$. This is now easy to do. We note that because B is a basis, the columns of P are linearly independent, the column rank of P is n and P is therefore invertible. We multiply (10) on the left by P^{-1} to get

$$\mathbf{x}' = P^{-1}\mathbf{x} \tag{11}$$

We note that P^{-1} is the transition matrix from E to B.

Then equation (11) (for $\mathbf{y}$) and equations (1) and (10) give

$$\mathbf{y}' = P^{-1}\mathbf{y} = P^{-1}A\mathbf{x} = P^{-1}AP\mathbf{x}'$$

Comparison with equation (8) shows that the new matrix D is given by

$$D = [T]_B = P^{-1}AP \tag{12}$$

Similarity The two matrices A and D both describe the same linear transformation T, using different coordinates.

DEFINITION 13 Two $n \times n$ matrices A and D are *similar* if there exists an invertible matrix P such that equation (12) holds. The matrix A is *diagonalizable* if it is similar to a diagonal matrix.

Eigenvectors We should choose a basis B of $\mathbf{R}^n$ that is related in some way to T. In the most favorable situation (which is not that rare), we can find a basis B of $\mathbf{R}^n$ that consists entirely of eigenvectors of T (or A). Specifically, we have

$$T(\mathbf{v}_i) = A\mathbf{v}_i = \lambda_i \mathbf{v}_i \tag{14}$$

for each i, where λ_i denotes the associated eigenvalue. On comparing equations (14) and (7), we deduce that the new matrix D is a *diagonal* matrix, with diagonal entries $d_{ii} = \lambda_i$ and $d_{ij} = 0$ for $i \neq j$.

THEOREM 15 (=Thm. 7.2.1) *The following conditions on the matrix A are equivalent:*

(a) *A is diagonalizable;*

(b) *$\mathbf{R}^n$ has a basis of eigenvectors of A;*

(c) *There is a basis B of $\mathbf{R}^n$ such that $D = [T]_B$ is a diagonal matrix.*

Proof We have just shown that (b) implies (a). The argument works just as well in reverse to show that (a) implies (b): if a matrix P exists as in equation (12) with D diagonal, we take $\mathbf{v}_j$ as the j-th column of P and then equation (7) shows that $\mathbf{v}_j$ is an eigenvector. Finally, equation (7) shows that (b) and (c) are equivalent. □