8-10 *Vector Fields and Fixed Points*

In this section we apply the fundamental group to two problems of geometry. One problem concerns the existence of vector fields tangent to given surfaces. The second deals with the "fixed-point problem": Given X, does every continuous map $f: X \longrightarrow X$ necessarily have a fixed point?

We shall obtain only some of the simpler results. Deeper theorems, including some of current research interest, demand much more of the machinery of algebraic topology than we have studied.

Theorem 10.1 *Given a nonvanishing vector field on* B^2, *there exists a point of* S^1 *where the vector field points directly inward and a point of* S^1 *where it points directly outward.*

Proof. A **vector field** on B^2 is an ordered pair $(x, v(x))$, where x is in B^2 and v is a continuous map of B^2 into R^2. In calculus, one often uses the notation

$$\mathbf{v}(x) = v_1(x)\mathbf{i} + v_2(x)\mathbf{j}$$

for the function v, where $\mathbf{i}$ and $\mathbf{j}$ are the standard unit basis vectors in R^2. But we shall stick with simple functional notation. To say that a vector field is *nonvanishing* means that $v(x) \neq \mathbf{0}$ for every x; in such a case v actually maps B^2 into $R^2 - \mathbf{0}$.

We show first that given v, it must point directly inward at some point of S^1.

Consider the map $w: S^1 \longrightarrow R^2 - \mathbf{0}$ obtained by restricting v to S^1. If there is no point x of S^1 at which the vector field points directly inward, then for no x in S^1 is $w(x)$ equal to a negative multiple of x. It follows that w is homotopic to the inclusion map $j: S^1 \longrightarrow R^2 - \mathbf{0}$, for the map $F: S^1 \times I \longrightarrow R^2 - \mathbf{0}$ given by the equation

$$F(x, t) = tx + (1 - t)w(x)$$

is the required homotopy. It is pictured in Figure 26. Obviously, F is continuous. To show that F never vanishes, note that if $F(x, t) = \mathbf{0}$, then

$$(1 - t)w(x) = -tx.$$

This equation is clearly false for $t = 0$ or $t = 1$, since $x \in S^1$ and $w(x) \neq \mathbf{0}$. For $0 < t < 1$, it says that $w(x) = -tx/(1 - t)$, so that $w(x)$ equals a negative multiple of x, which is forbidden.

Since w is homotopic to the inclusion map $j: S^1 \longrightarrow R^2 - \mathbf{0}$, it must be essential. On the other hand, w is extendable to the continuous map $v: B^2 \longrightarrow R^2 - \mathbf{0}$, so that it is inessential. Thus we arrive at a contradiction. Therefore, v points directly inward at some point of S^1.

Consider now the nonvanishing vector field $(x, -v(x))$. By the result just

Theorem 10.4. *The sphere* S^2 *possesses no nonvanishing tangent vector field.*

Proof. Suppose that S^2 had a nonvanishing vector field $(x, v(x))$. Consider the north pole $p = (0, 0, 1)$ of S^2; we shall assume for convenience that at p the vector field is parallel to the y-axis. Take a small open ball U in S^2 centered at p, so small that on the ball U the vector field does not vary more than a few degrees from being parallel to the y-axis.

Now consider the map $f: S^2 - p \longrightarrow R^2$ given by "stereographic projection." (See Step 1 of the proof of Theorem 6.2.)

The map f is, in fact, a homeomorphism of $S^2 - p$ with R^2. More than that, it carries *tangent vectors* to S^2 continuously into *tangent vectors* to R^2. How? The easiest way to see what happens is to take a given tangent vector v at a point x, and to find a curve C in S^2 having that vector as its velocity vector at x. The map f carries the curve C into a curve in R^2, and we let it take the vector v into the velocity vector of the image curve $f(C)$ at the point $f(x)$. Let us denote the image of (x, v) by (y, w). You can check by direct computation that f is *smooth* (w is a well-defined, continuous function of (x, v)) and *nonsingular* (w is nonzero if v is nonzero).

Now we ask the question: What happens to the nonvanishing vector field (x, v) under this map? It is carried into a nonvanishing vector field (y, w) on R^2. In particular, consider the subspace $S^2 - U$ of S^2, which the

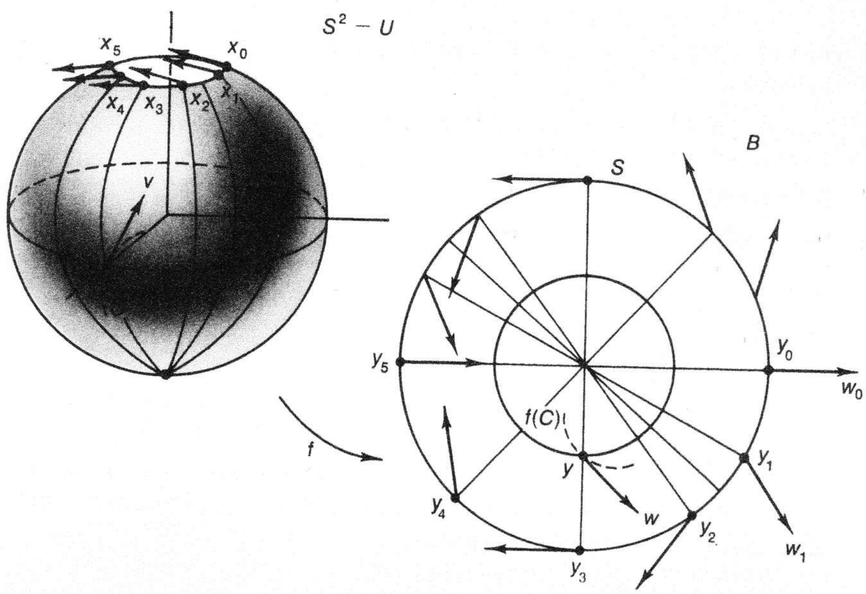

Figure 28

map f carries onto a ball B in R^2 of large radius about the origin. What does the image of the vector field look like? We have sketched in Figure 28 what it looks like on the large circle S that is the boundary of B.

So far so good. But let us examine this vector field $(y, w(y))$ on B more closely, particularly its restriction to the circle S. You can see from the picture that the map $h : S \longrightarrow R^2 - \mathbf{0}$ defined by

$$h(y) = w(y)$$

carries a generator of $\pi_1(S, x_0)$ to twice a generator of $\pi_1(R^2 - \mathbf{0}, x_0)$. Intuitively, as y goes around the circle S once, the point $h(y)$ goes around the origin twice.

Now comes the contradiction. We know that h is extendable to a map of B into $R^2 - \mathbf{0}$, because $(y, w(y))$ is a nonvanishing vector field on B. Thus h_* must be the zero homomorphism of fundamental groups. On the other hand, we know both fundamental groups in question are infinite cyclic groups, and that h_* carries a generator of the first to twice a generator of the second. In particular, h_* is *not* the zero homomorphism. □

These theorems have generalizations to higher dimensions, which are discussed in Exercises 7 and 8.

Exercises

1. Show that if A is a retract of B^2, then every continuous map $f: A \longrightarrow A$ has a fixed point.

2. Show that if A is a nonsingular 3 by 3 matrix having nonnegative entries, then A has a positive real eigenvalue.

3. Show that the set B of Corollary 10.3 is homeomorphic to B^2.

***4.** Try to give an algebraic proof of Corollary 10.3. (This exercise is for those students who are tempted by the thought that this result is trivial! You might try the 2 by 2 case first.)

5. Show that if $f: S^1 \longrightarrow S^1$ is inessential, then f has a fixed point and f carries some point x to its antipode $-x$.

6. Show that given a continuous map $f: S^2 \longrightarrow S^2$, either f has a fixed point or f carries some point x to its antipode $-x$. [*Hint:* Use Theorem 10.4.]

7. Suppose that you are given the fact that for each n, there is no retraction $r : B^{n+1} \longrightarrow S^n$. (This result can be proved using techniques of algebraic topology). Derive the following corollaries:
(a) The identity map $i : S^n \longrightarrow S^n$ is essential.
(b) The antipodal map $a : S^n \longrightarrow S^n$ is essential.
(c) The inclusion map $j : S^n \longrightarrow R^{n+1} - \mathbf{0}$ is essential.