

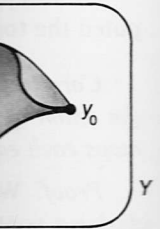
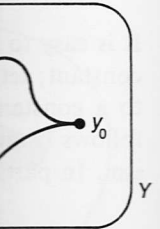
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4. Recall that a space Y is said to be *contractible* if the identity map $i_Y : Y \rightarrow Y$ is inessential. Show that if Y is contractible, then Y is simply connected.
5. A map $h : S^n \rightarrow S^m$ is said to be **antipode-preserving** if $h(-x) = -h(x)$ for every $x \in S^n$.

Theorem. If $h : S^1 \rightarrow S^1$ is antipode-preserving and continuous, then h is essential.

- (a) Let $p : S^1 \rightarrow S^1$ be the map $p(z) = z^2$, where z is a complex number. Show that h induces a continuous map $g : S^1 \rightarrow S^1$ such that $p \circ h = g \circ p$.
- (b) Show that if f is any path in S^1 from a point x to its antipode $-x$, then $p \circ f$ is a loop in S^1 that is not path homotopic to a constant.
- (c) Show that both p and g induce monomorphisms (injective homomorphisms) of the fundamental groups.
- (d) Conclude that h is essential.

6. Assume Exercise 5.

- (a) Prove the following:

Theorem (Borsuk-Ulam theorem for S^2). There is no continuous antipode-preserving map $f : S^2 \rightarrow S^1$.

[Hint: Consider the equator in S^2 .] It is in general true that there is no continuous antipode-preserving map $f : S^n \rightarrow S^m$ if $m < n$. But the proof requires more tools than we now possess.

- (b) Prove:

Theorem. Given a continuous map $f : S^2 \rightarrow R^2$, there is a point x of S^2 such that $f(x) = f(-x)$.

[Hint: Consider $(f(x) - f(-x)) / \|f(x) - f(-x)\|$.]

- (c) Prove:

Theorem (A "theorem of meteorology"). At any given moment in time, there exists a pair of antipodal points on the surface of the earth at which both the temperature and the barometric pressure are equal.

- (d) Prove:

Theorem. If $g : S^2 \rightarrow S^2$ is continuous and $g(x) \neq g(-x)$ for all x , then g is surjective.

- *7. Generalize Exercise 5 as follows: Assume that $h : S^1 \rightarrow S^1$ is antipode-preserving; let $h(x_0) = x_1$. Then h_* maps a generator of $\pi_1(S^1, x_0)$ to an *odd* multiple of a generator of $\pi_1(S^1, x_1)$.

8-9 The Fundamental Theorem of Algebra

It is a basic fact about the complex numbers that every polynomial equation

$$x^n + a_{n-1}x^{n-1} + \dots + a_1x + a_0 = 0$$

of degree n with real or complex coefficients has n roots (if the roots are counted according to their multiplicities). You probably first were told this fact in high school algebra, although it is doubtful that it was proved for you at that time.