Final Exam, Linear Algebra, Spring, 2003, W. Stephen Wilson

Name: Solutions

TA Name and section: ______________________

NO CALCULATORS.

(1) (3 points) Give a definition of *linear independence.*

$u_1, u_2, \ldots, u_n$ are l. i. if

$a_1 u_1 + a_2 u_2 + \cdots + a_n u_n = 0$

implies all $a_i = 0$

(2) (3 points) If $A : \mathbb{R}^n \to \mathbb{R}^n$, there is a relationship between the dimension of the kernel of A and the dimension of the image of A. What is it?

$n = \dim \operatorname{Ker} A + \dim \operatorname{Image} A$

(3) (3 points) Define *perpendicular* in $\mathbb{R}^n$.

$x, y \in \mathbb{R}^n$ are perpendicular

if $x \cdot y = 0$

(4) (3 points) Define the *length* of a vector in $\mathbb{R}^n$.

length $u = \sqrt{u \cdot u}$

(5) (3 points) Find a basis for the orthogonal complement (in $\mathbb{R}^4$) of the subspace spanned by $\underset{u_1}{\begin{pmatrix} 1 \\ 0 \\ 2 \\ 3 \end{pmatrix}}$ and $\underset{u_2}{\begin{pmatrix} 0 \\ 1 \\ 1 \\ 2 \end{pmatrix}}$.

$V^\perp$ is all x such that $u_1 \cdot x = 0$ and $u_2 \cdot x = 0$ i.e.

$1 \cdot x_1 + 0 \cdot x_2 + 2x_3 + 3x_4 = 0$

$0\, x_1 + 1 \cdot x_2 + 1 \cdot x_3 + 2x_4 = 0$

matrix $\begin{matrix} 1 & 0 & 2 & 3 \\ 0 & 1 & 1 & 2 \end{matrix}$

Solve, row reduced already

so $x_1 = -2x_3 - 3x_4$ $x_2 = -x_3 - 2x_4$

~~x_3~~ $x_3 = s$ $x_4 = r$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = s \begin{pmatrix} -2 \\ -1 \\ 1 \\ 0 \end{pmatrix} + r \begin{pmatrix} -3 \\ -2 \\ 0 \\ 1 \end{pmatrix}$$

Basis

We will now be working with the matrix $A = \begin{pmatrix} 3 & 0 & -3 \\ -1 & 4 & 9 \\ 0 & -2 & -4 \end{pmatrix}$ for some time now.

(6) (3 points) What is the trace of A above?

trace $= 3+4-4=3$

(7) (3 points) What is the determinant of A above?

$3\cdot 4\cdot(-4)+(-3)(-1)(-2)-(-2)(9)(3)$

$= -48-6+54=0$

(8) (3 points) Solve the equation $Ax = 0$ for the A above.

$$\begin{matrix} 3 & 0 & -3 \\ -1 & 4 & 9 \\ 0 & -2 & -4 \end{matrix} \rightarrow \begin{matrix} 1 & 0 & -1 \\ 0 & 4 & 8 \\ 0 & -2 & -4 \end{matrix}$$

$$\begin{matrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{matrix}$$

$x_1 = x_3 \quad x_2 = -2x_3 \quad x_3 = s$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = s \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$ solution.

(9) (3 points) For the A above, solve the equation $Ax = \begin{pmatrix} 0 \\ 8 \\ -4 \end{pmatrix}$.

$$\begin{matrix} +3 & 0 & -3 & 0 \\ -1 & 4 & 9 & 8 \\ 0 & -2 & -4 & -4 \end{matrix} \quad \begin{matrix} 1 & 0 & -1 & 0 \\ 0 & 4 & 8 & 8 \\ 0 & -2 & -4 & -4 \end{matrix} \rightarrow \begin{matrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 \end{matrix}$$

$x_1 = x_3 \quad x_2 = -2x_3 + 2 \quad x_3 = s$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = s \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}$$ Solution

(10) (3 points) Find a basis for the kernel of the A above.

From # 8 $\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$

(11) (3 points) Find a basis for the row space of the A above.

From # 8 $(1, 0, -1)$

and $(0, 1, 2)$

(12) (3 points) What is the rank of the A above?

From #8 2

(13) (3 points) Find a basis for the image of the A above.

From #8 we can use the first and second columns.

$$\begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix} \ \& \ \begin{pmatrix} 0 \\ 4 \\ -2 \end{pmatrix}$$

(14) (3 points) Find the characteristic polynomial for the A above.

$$\text{char poly} = \cancel{\lambda I} \det(\lambda I - A) = \det\begin{pmatrix} \lambda-3 & 0 & 3 \\ 1 & \lambda-4 & -9 \\ 0 & 2 & \lambda+4 \end{pmatrix}$$

$$= (\lambda-3)(\lambda-4)(\lambda+4) + 3\cdot 1\cdot 2 - 2\cdot(-9)(\lambda-3)$$

$$= (\lambda-3)(\lambda^2-16) + 6 + 18\lambda - 54$$

$$= \cancel{\lambda^3 - 19\lambda + 48 + 6 + 18\lambda}$$

$$= \lambda^3 - 3\lambda^2 - 16\lambda + \cancel{54}\,48 + 6 + 18\lambda - 54$$

$$= \lambda^3 - 3\lambda^2 + 2\lambda + 0 = \lambda(\lambda-1)(\lambda-2)$$

$$= \lambda^3 - 3\lambda^2 + 2\lambda$$

(15) (3 points) Find the Eigenvalues for the A above. (They are all integers.)

From 14 $\quad \lambda = 0, 1, 2$

(16) (3 points) For the A above, find an Eigenvector for each of the Eigenvalues. To make it easier to grade, choose Eigenvectors with integer coordinates where the integers are as small as possible.

For 0 we already have $\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$ #8

For 1

$$\begin{matrix} -2 & 0 & 3 \\ 1 & -3 & -9 \\ 0 & 2 & 5 \end{matrix} \rightarrow \begin{matrix} 1 & 0 & -3/2 \\ 0 & -3 & -7\frac{1}{2} \\ 0 & 1 & 5/2 \end{matrix}$$

$$\rightarrow \begin{matrix} 1 & 0 & -3/2 \\ 0 & 1 & 5/2 \\ \cancel{0} & \cancel{1} & \cancel{5/2} \end{matrix} \qquad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3 \\ -5 \\ 2 \end{pmatrix}$$

For 2

$$\begin{matrix} -1 & 0 & +3 \\ +1 & -2 & -9 \\ 0 & 2 & 6 \end{matrix} \rightarrow \begin{matrix} 1 & 0 & -3 \\ 0 & -2 & \cancel{+2}\,-6 \\ 0 & 2 & 6 \end{matrix}$$

$$\begin{matrix} 1 & 0 & -3 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{matrix} \qquad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3 \\ -3 \\ 1 \end{pmatrix}$$

(17) (3 points) Use your Eigenvectors to make a basis of $\mathbb{R}^3$. Chose the first basis vector to be the Eigenvector associated with the largest Eigenvalue and the third basis vector to be the Eigenvector associated with the smallest Eigenvalue. Call this basis $\mathcal{B}$. We have a linear transformation given to us by A. What is the matrix B when we use coordinates from this new basis, i.e. $B : [x]_{\mathcal{B}} \to [Ax]_{\mathcal{B}}$?

$$\mathcal{B} = \left\{ \begin{pmatrix} 3 \\ -3 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ -5 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \right\}$$

$$B = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Leftarrow \text{Solution}$$

(18) (3 points) We know there is a matrix S such that $S^{-1}AS = B$. Find S.

$$S = \begin{pmatrix} 3 & 3 & 1 \\ -3 & -5 & -2 \\ 1 & 2 & 1 \end{pmatrix} \quad \text{from } 16 \,\&\, 17$$

(19) (3 points) Find S^{-1}.

$$\left(\begin{array}{ccc|ccc} 3 & 3 & 1 & 1 & 0 & 0 \\ -3 & -5 & -2 & 0 & 1 & 0 \\ 1 & 2 & 1 & 0 & 0 & 1 \end{array}\right) \rightarrow \left(\begin{array}{ccc|ccc} 1 & 1 & 1/3 & 1/3 & 0 & 0 \\ 0 & -2 & -1 & 1 & 1 & 0 \\ 0 & 1 & 2/3 & -1/3 & 0 & 1 \end{array}\right)$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & -1/3 & 2/3 & 0 & -1 \\ 0 & 1 & 2/3 & -1/3 & 0 & 1 \\ \cancel{0} & \cancel{0} & \cancel{1/3} & \cancel{1/3} & \cancel{1} & \cancel{2} \\ 0 & 0 & 1 & 1 & 3 & 6 \end{array}\right) \qquad \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & -1 & -2 & -3 \\ 0 & 0 & 1 & 1 & 3 & 6 \end{array}\right)$$

$$S^{-1} = \begin{pmatrix} 1 & 1 & 1 \\ -1 & -2 & -3 \\ 1 & 3 & 6 \end{pmatrix} \longleftarrow \text{answer}$$

check $\left(\downarrow \right) \times \begin{pmatrix} 3 & 3 & 1 \\ -3 & -5 & -2 \\ 1 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

(20) (3 points) Let P_3 be all polynomials with degree less than or equal to 3. Let $C[-1,1]$ be the continuous functions from the interval $[-1,1]$ to the reals. $P_3 \subset C[-1,1]$. They have an inner product given by $< f, g >= \int_{-1}^{1} f(x)g(x)dx$. Show the set $V \subset P_3 \subset C[-1,1]$ defined as all $f \in P_3$ with $f(-1) = 0$ and $\int_{-1}^{1} f(x)dx = 0$ is a linear subspace of P_3.

Show $f+g \in V$ if f & g are. also $kf \in V$

$(f+g)(-1) = f(-1) + g(-1) = 0 + 0$

$kf(-1) = k(f(-1)) = k \cdot 0 = 0$

$\int_{-1}^{1}(f+g)\,dx = \int_{-1}^{1} f\,dx + \int_{-1}^{1} g\,dx = 0 + 0 = 0$

$\int_{-1}^{1} kf\,dx = k\int_{-1}^{1} f\,dx = k \cdot 0 = 0$

(21) (3 points) Find a basis for the linear space in (20).

arbitrary $f \in P_3$ $\quad ax^3 + bx^2 + cx + d.$

if $0 = f(-1) = -a + b - c + d$

& $0 = \int_{-1}^{1} f\,dx = \left[\frac{ax^4}{4} + \frac{bx^3}{3} + \frac{cx^2}{2} + dx\right]_{-1}^{1} = \frac{2b}{3} + 2d$

$\underline{b = -3d}$ $\quad$ ~~$d = a - b + c$~~

$0 = -a - 3d - c + d = -a - c - 2d$

$\underline{a = -c - 2d}$

$f = (-c-2d)x^3 - 3dx^2 + cx + d$

$= c(-x^3 + x) + d(-2x^3 - 3x^2 + 1)$

basis $\quad -x^3 + x$, $-2x^3 - 3x^2 + 1$ $\quad$ or $\quad$ or other

although I prefer $\quad x^3 - x$, $3x^2 + 2x^3 - 1$?

ex 2 p. 3

We work in $P_1 \subset P_2 \subset C[0,1]$ with the usual inner product ($< f, g >= \int_0^1 f(x)g(x)dx$). Recall that P_n is the set of polynomials of degree less than or equal to n. It is a fact that an orthonormal basis for P_1 is given by $\{1, \sqrt{3}(2x-1)\}$. You can now assume that.

(22) (3 points) What is the orthogonal projection of $x^2 \in P_2$ onto $P_1 \subset P_2$, i.e. $\text{proj}_{P_1}(x^2)$? (Show work.)

$u_1 = 1 \qquad u_2 = \sqrt{3}(2x-1)$

$\text{proj}_{P_1}(x^2) = \cancel{<u_1, x^2> 1 + <u_2}$

$= <u_1, x^2> u_1 + <u_2, x^2> u_2$

$= \int_0^1 x^2 dx + \int_0^1 x^2 (2\sqrt{3}x - \sqrt{3}) dx \left(\sqrt{3}(2x-1)\right)$

$= \left.\frac{x^3}{3}\right]_0^1 + 3\left(\int_0^1 x^2(2x-1)dx\right)(2x-1)$

$= \frac{1}{3} + 3\left(\int_0^1 (2x^3 - x^2)dx\right)(2x-1)$

$= \frac{1}{3} + 3\left[\frac{2x^4}{4} - \frac{x^3}{3}\right]_0^1 (2x-1)$

$= \frac{1}{3} + 3\left[\frac{1}{2} - \frac{1}{3}\right](2x-1)$

$= \frac{1}{3} + 3\left(\frac{1}{6}\right)(2x-1) = \frac{1}{3} + \frac{1}{2}(2x-1)$

$= \frac{1}{3} + x - \frac{1}{2} = \boxed{x - \frac{1}{6}}$

(23) (3 points) Using the basis $\{x^2, x, 1\}$ for P_2, find the 3×3 matrix for $\text{proj}_{P_1} : P_2 \to P_1 \subset P_2$.

$$\text{proj}_{P_1}(x^2) = 0x^2 + 1 \cdot x - \frac{1}{6}$$
$$\text{proj}_{P_1}(x) = 0x^2 + 1 \cdot x + 0$$
$$\text{proj}_{P_1}(1) = 0x^2 + 0x + 1$$

Matrix is $\left(\text{proj}_{P_1}(x^2), \text{proj}_{P_1}(x), \text{proj}_{P_1}(1)\right)$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ -1/6 & 0 & 1 \end{pmatrix}$$

ex 2 p. 4

(24) (3 points) What is the dimension of the kernel of this linear transformation (in problem (23))? (Explain.)

rank = 2 so kernel has dim = 1.

(25) (3 points) Find a polynomial basis for the kernel in (24).

$x^2 \to x - 1/6$

$x - 1/6 \to x - 1/6$

So $x^2 - x + 1/6 \in \mathrm{Ker}$

(26) (3 points) Find an orthogonal (not necessarily orthonormal) basis for P_2 that extends the known orthogonal basis for P_1 from before problem (22).

the kernel of proj_{P_1} has basis $x^2 - x + 1/6$.

the kernel is $\perp$ to P_1 so

$1, \sqrt{3}(2x-1), x^2 - x + 1/6$

(27) (3 points) The polynomial of problem (22) minimizes a certain *least squares* integral. What is that integral?

$ax + b = x - 1/6$

minimizes

$$\int_0^1 (x^2 - (ax+b))^2 \, dx$$

(28) (3 points) For the equation $\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, find all of the least squares solutions.

$\begin{matrix} 1 & 2 & 1 \\ 2 & 4 & 0 \end{matrix} \to \begin{matrix} 1 & 2 & 1 \\ 0 & 0 & -2 \end{matrix}$ inconsistent

need to project $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ onto image of $\begin{matrix} 1 & 2 \\ 2 & 4 \end{matrix} \to \begin{matrix} 1 & 2 \\ 0 & 0 \end{matrix}$

basis of image is first column $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$

make it length 1, get $\frac{1}{\sqrt{5}}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = u$

$\text{proj.}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \left\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix}, u \right\rangle u = \frac{1}{\sqrt{5}} \cdot \frac{1}{\sqrt{5}}\left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix}\right]\begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$= \frac{1}{5} \cdot 1 \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 1 \\ 2 \end{pmatrix}$

solve this $\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$\begin{matrix} 1 & 2 & 1/5 \\ 2 & 4 & 2/5 \end{matrix} \quad \begin{matrix} 1 & 2 & 1/5 \\ 0 & 0 & 0 \end{matrix}$

$x_1 + 2x_2 = 1/5$

$x_1 = 1/5 - 2x_2$

$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = s\begin{pmatrix} -2 \\ 1 \end{pmatrix} + \begin{pmatrix} 1/5 \\ 0 \end{pmatrix}$

ex2 p5

(29) (3 points) If $\det(A) = 5$ and A is an $n \times n$ matrix, then what is $\det(5A)$? (Explain a bit.)

det is linear in each column so we can take the 5 out n times.

$$\det(5A) = 5^n \det A = 5^{n+1}$$

(30) (3 point) If A is an orthogonal rotation $n \times n$ matrix, then what is $\det(5A)$? (Explain a bit.)

$\det A = 1$

$\det 5A = 5^n \det A = 5^n$ as above.

(31) (3 points) Write the quadratic form x_1x_2 in matrix form with a symmetric matrix.

$$\begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix}$$

(32) (3 points) Say whether $x_1x_2 = 1$ is an ellipse or a hyperbola. Justify using the approach in the course.

Find Eigenvalues

$$\det\begin{pmatrix} \lambda & -1/2 \\ -1/2 & \lambda \end{pmatrix} = \lambda^2 - \left(\frac{1}{2}\right)^2 = 0$$

$$\lambda^2 = \left(\frac{1}{2}\right)^2 \qquad \lambda = \pm\frac{1}{2}$$

one positive one negative

hyperbola

(33) (3 points) Find the principal axes for $x_1 x_2 = 1$ and locate the intercepts.

Find Eigenvectors

$$\begin{pmatrix} \frac{1}{2} & -1/2 \\ -1/2 & 1/2 \end{pmatrix} \rightarrow \begin{matrix} 1 & -1 \\ 0 & 0 \end{matrix} \qquad x_1 = x_2$$

Eigenvector. $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

for $\lambda = 1/2$

for $\lambda = -1/2$ $\quad \begin{matrix} -1/2 & -1/2 \\ -1/2 & -1/2 \end{matrix} \rightarrow \begin{matrix} 1 & 1 \\ 0 & 0 \end{matrix}$ get $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

for $\lambda = -1/2$

Normalize to get $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ & $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

use this basis. in new coordinates.

$$1 = x_1 x_2 = \lambda_1 c_1^2 + \lambda_2 c_2^2 = \frac{1}{2} c_1^2 \cancel{+} - \frac{1}{2} c_2^2$$

intercept is when $c_2 = 0$ then $1 = \frac{1}{2} c_1^2$

$$c_1^2 = 2 \qquad c_1 = \pm\sqrt{2}$$

so intercept is at. $\sqrt{2} \cdot \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

and $-\sqrt{2} \cdot \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = -\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

(34) (3 points) Give the form of the curve $x_1 x_2 = 1$ in the coordinate system defined by the principal axes.

From 33

$$\frac{1}{2} c_1^2 - \frac{1}{2} c_2^2 = 1$$

(35) (3 points) State Cramer's rule.

See book

ex2 p.5

(36) (3 points) What is the area of the parallelogram defined by the vectors $(1,5)$ and $(3,9)$.

$$\left|\det\begin{pmatrix}1 & 3\\ 5 & 9\end{pmatrix}\right| = |9-15| = |-6| = 6$$

(37) (3 points) If A is invertible, what is A^{-1} in terms of the adjoint (which you should define of course)?

See book.

(38) (3 points) We will now study the matrix $A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 1 & 0 \end{pmatrix}$. The book tells us that there is an orthonormal basis $\{v_1, v_2\}$ for $\mathbb{R}^2$ and an orthonormal basis $\{u_1, u_2, u_3\}$ for $\mathbb{R}^3$ such that $Av_1 = \sigma_1 u_1$ and $Av_2 = \sigma_2 u_2$ (with $\sigma_1 \geq \sigma_2$). Find σ_1 and σ_2.

σ_1 & σ_2 are the square roots of the Eigen values of $A^T A$

$$= \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

$$\det\begin{pmatrix} \lambda-2 & -1 \\ -1 & \lambda-2 \end{pmatrix} = (\lambda-2)^2 - 1 = \lambda^2 - 4\lambda + 3 = (\lambda-3)(\lambda-1)$$

$\lambda = 3, 1$ $\quad \sigma_1 = \sqrt{3}$ $\quad \sigma_2 = 1$

(39) (3 points) Find v_1 and v_2 in the previous problem.

v_1 & v_2 are the Eigenvectors of A^TA

So $A^TA = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ for $\lambda = 3$

$\begin{matrix} 1 & -1 \\ -1 & 1 \end{matrix} \rightarrow \begin{matrix} 1 & -1 \\ 0 & 0 \end{matrix}$ $x_1 = x_2$ $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ normal. $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = v_1$

for $\lambda = 1$ $\begin{matrix} -1 & -1 \\ -1 & -1 \end{matrix} \rightarrow \begin{matrix} 1 & 1 \\ 0 & 0 \end{matrix}$ $x_1 = -x_2$ $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = v_2$

(40) (3 points) Find u_1, u_2, and u_3 in the previous problems.

$A v_1 = \sigma_1 u_1 \qquad A v_2 = \sigma_2 u_2$

$$\begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 1 & 0 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = \sqrt{3} \underbrace{\frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}}_{u_1}$$

$A v_2 = \sigma_2 u_2$

$$\begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 1 & 0 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = 1 \underbrace{\frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}}_{u_2}$$

so $u_1 = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ these are orthonormal

$u_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

need to find u_3 such that $u_1 \cdot u_3 = 0 = u_2 \cdot u_3$

so $x_1 + 2x_2 + x_3 = 0 \rightarrow \begin{matrix} 1 & 2 & 1 \\ -1 & 0 & 1 \end{matrix} \rightarrow \begin{matrix} 1 & 2 & 1 \\ 0 & 2 & 2 \end{matrix} \rightarrow \begin{matrix} 1 & 2 & 1 \\ 0 & 1 & 1 \end{matrix} \rightarrow \begin{matrix} 1 & 0 & -1 \\ 0 & 1 & 1 \end{matrix}$

$-x_1 \qquad + x_3 = 0$

$x_1 = \cancel{}\, x_3$ so

$x_2 = -x_3$

$$\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \frac{1}{\sqrt{3}} = u_3$$