The Homology of the Double Loop Space of the Thom Space $MU(n)$

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§0. Introduction

In this paper, we calculate the homology of the double loop space of the Thom space of the classifying space for complex n-plane bundles with coefficients F_2, $n > 1$, $H_*(\Omega^2 MU(n); F_2)$, where F_2 is the field with 2 elements. The main result is as follows.

Theorem 6.9 *$H_*(\Omega^2 MU(n); F_2)$ is a polynomial algebra. The module of generators $QH_*(\Omega^2 MU(n); F_2)$ has a basis isomorphic to $\{[\![e'_1]\!], [\![e'_2]\!], \ldots\}$, where $\{e'_1, e'_2, \ldots\}$ is a basis of the primitive module $P\mathrm{Cotor}^{H_*(MU(n);F_2)}(F_2, F_2)$ and $\deg[\![e'_i]\!] = \deg e'_i - 1$.*

We consider the following natural map $f: S^2 MU(n-1) \longrightarrow MU(n)$. It then induces an injective map $(\Omega f)^*: H^*(\Omega MU(n); F_2) \to H^*(\Omega S^2 MU(n-1); F_2)$ (See Proposition 3.5). Therefore we expecte to study $H(\Omega MU(n); F_2)$ and $H(\Omega^2 MU(n); F_2)$ by studying $H(\Omega S^2 MU(n-1); F_2)$ and $H(\Omega^2 S^2 MU(n-1); F_2)$. Using the Eilenberg-Moore spectral sequence, we obtain that $H^*(\Omega S^2 MU(n-1); F_2)$ *is an exterior algebra* (See Theorem 2.9). Hence $H^*(\Omega MU(n); F_2)$ *is an exterior algebra* (See Theorem 3.7).

In order to calculate $H^*(\Omega^2 MU(n); F_2)$ by Eilenberg-Moore spectral sequence, we need to obtain generators of $H^*(\Omega MU(n); F_2)$. We notice that since $H_*(\Omega S^2 MU(n-1); F_2)$ is the tensor algebra $T(\sum_{q>0} H_q(SMU(n-1); F_2))$ (see [3]), the primitive module of $H_*(\Omega S^2 MU(n-1); F_2)$ is the free restricted Lie algebra on $H_*(SMU(n-1); F_2)$ (See Proposition 5.6). So we consider the dual of our result and obtain that the map $\mathrm{Cotor}^{H_*(S^2 MU(n-1);F_2)}(F_2, F_2) \to \mathrm{Cotor}^{H_*(MU(n);F_2)}(F_2, F_2)$ is surjective and that the kernel of the map is the ideal in $T(\sum_{q>0} H_q(SMU(n-1); F_2))$ generated by

$$\{ \sum_{i_1=1,\ldots,i_{n-1}=1}^{m_1-1,\ldots,m_{n-1}-1} g_{i_1,\ldots,i_{n-1}} \otimes g_{m_1-i_1,\ldots,m_{n-1}-i_{n-1}} : m_j > 1, \text{ for all } j\}$$

(See theorem 4.3), where $g_{j_1,\ldots,j_{n-1}} = s^{-1}(b_{j_1} \circ \ldots \circ b_{j_{n-1}})$, $b_{j_1} \circ \ldots \circ b_{j_{n-1}}$ is the basis of $H_*(MU(n-1); F_2)$ and S is the suspension isomorphism $S_p : H_p(SMU(n-1); F_2) \to H_{p-1}(MU(n-1); F_2)$. Denote $H' = \mathrm{Cotor}^{H_*(MU(n);F_2)}(F_2, F_2)$. We have *the homomorphism $H_*(\Omega S^2 MU(n-1); F_2) \to H'$ induces a surjective homomorphism on their primitive module $PH_*(\Omega S^2 MU(n-1); F_2) \to PH'$* (See Theorem 5.7). Furthermore *the primitive* *module PH' is spanned by $LG' \cup \{g_{1,\ldots,1}\}$ as a vector space, where LG' is the restricted Lie algebra on G' and G' is a vector space spanned by $\{g_{i_1,\ldots,i_{n-1}} : 0 < i_1 \leq \ldots \leq i_{n-1}, i_{n-1} > 1\}$* (See Theorem 5.12).

We denote by EPH' the exterior algebra on PH', which is isomorphic to $H_*(\Omega MU(n); F_2)$ as a coalgebra by the Poincaré-Birkhoff-Witt Theorem. Since the computation of the homology of the Eilenberg-Moore spectral sequence only needs the coalgebra structure, we can show the theorem 6.9.

This paper is an extended version of the author's thesis. The author wishes to thank her advisors W.S. Wilson and J. Boardman for their guidance and very helpful conversation.

§1. The Eilenberg-Moore Spectral Sequence

In the remaining sections, K always denotes the field F_2, and we abbreviate $H^*(\ ;K)$ to $H^*(\ )$, etc.

Let A be a differential graded augmented algebra over K with differential ∂. Define $B^{-k}(A,K) = A \otimes \underbrace{\widetilde{A} \otimes \cdots \otimes \widetilde{A}}_{k}$, where $\widetilde{A} = \mathrm{Ker}[\varepsilon : A \to K]$. It is customary to denote $B^{-k}(A,K)$ as B^{-k}, a generating element of B^{-k} as $a_0[a_1|a_2|\cdots|a_k]$, and an element of B^0 as $a_0[\,]$. Define $\delta : B^{-k} \to B^{-k+1}$, $\partial : B^{-k,n} \to B^{-k,n+1}$ and $\varepsilon : B^0 \to K$, as follows,

$$\delta(a_0[a_1|\cdots|a_k]) = a_0a_1[a_2|\cdots|a_k] + \sum_{i=1}^{k-1} a_0[a_1|\cdots|a_ia_{i+1}|\cdots|a_k],$$

$$\partial(a_0[a_1|\cdots|a_k]) = (\partial a_0)[a_1|\cdots|a_k] + \sum_{i=1}^{k} a_0[a_1|\cdots|\partial a_i|\cdots|a_k]$$

and

$$\varepsilon(a_0[\,]) = \varepsilon(a_0).$$

We can check that $\delta\delta = 0, \partial\partial = 0, \partial\delta + \delta\partial = 0$ and that δ is an A-morphism.

Definition 1.1 $\mathrm{Tor}_A(K,,K) = H(\overline{B}^\bullet, D)$. where $\overline{B}^\bullet = K \otimes_A B^\bullet$ and the K-graded differential module $(B^\bullet, D)$ formed by letting $(B^\bullet)^j = \bigoplus_{n+m=j} (B^m)^n$, $D = \delta + \partial$.

Let Σ_{p+q} be the group of permutations of the set $\{1, 2, \ldots, p+q\}$. Any $\sigma \in \Sigma_{p+q}$ is called a (p,q)-*shuffle* if the following hold:

$$\sigma(1) < \sigma(2) < \ldots < \sigma(p),$$
$$\sigma(p+1) < \sigma(p+2) < \ldots < \sigma(p+q).$$

Suppose

$$[a_1|a_2|\cdots|a_k] \in \overline{B}^{-k}, \quad [b_1|b_2|\cdots|b_s] \in \overline{B}^{-s}.$$

Define the *shuffle product*

$$*: \quad \overline{B}^{-k} \otimes \overline{B}^{-s} \to \overline{B}^{-k-s}$$

by

$$[a_1|a_2|\cdots|a_k] * [b_1|b_2|\cdots|b_s] = \sum_{(k,s)-\text{shuffles } \sigma} [w_{\sigma^{-1}(1)}|w_{\sigma^{-1}(2)}|\cdots|w_{\sigma^{-1}(k+s)}],$$

where $w_i = a_i$ for $i \leq k$, $w_j = b_{j-k}$ for $k < j \leq k+s$. The shuffle product induces a multiplication

$$*: \quad \mathrm{Tor}_A(K,K) \otimes \mathrm{Tor}_A(K,K) \to \mathrm{Tor}_A(K,K).$$

We also can define the coproduct

$$\Delta: \quad \overline{B}^{-n} \to \bigoplus_{r+s=n} \overline{B}^{-r} \otimes \overline{B}^{-s}$$

on a typical element, $[a_1|a_2|\cdots|a_n]$, by

$$\Delta[a_1|a_2|\cdots|a_n] = \sum_{j=0}^{n} [a_1|a_2|\cdots|a_j] \otimes [a_{j+1}|a_{j+2}|\cdots|a_n].$$

Theorem 1.2 *If A is a graded differential algebra over K, then with the $*$ multiplication and Δ comultiplication, $\overline{B}^{\bullet}$ is a differential Hopf algebra and this induces the structure of Hopf algebra on $\mathrm{Tor}_A(K,K)$ with multiplication, $*$, as given above.*

Proof See [1] p241 (7.15).

Theorem 1.3 (Eilenberg-Moore) *There is a spectral sequence, lying in the second quadrant, with $E_2^{*,*} = \mathrm{Tor}^{*,*}_{H(A)}(K,K)$, which converges to $\mathrm{Tor}^{*,*}_A(K,K)$, where A is a differential graded algebra over K.*

Proof See [1] p226 (7.5).

Theorem 1.4 *The Eilenberg-Moore spectral sequence*

$$\mathrm{Tor}_{H(A)}(K,K) \Longrightarrow \mathrm{Tor}_A(K,K)$$

is a spectral sequence of Hopf algebras, converging to its target as a Hopf algebra. The natural Hopf algebra structure on $\mathrm{Tor}_{H(A)}(K,K)$ *agrees with the Hopf algebra structure induced by the spectral sequence.*

Proof Since the spectral sequence is induced by the filtration $F^{-k}(\overline{B}^{\bullet}) = \bigoplus_{-s \geq -k} \overline{B}^{-s}$, it is easy to check that the multiplication and comultiplication are compatible with the filtration. Since $\frac{F^{-s}}{F^{-s+1}} = \overline{B}^{-s}$, we have

$$E_1^{-s,*} = H^{-s+*}(\frac{F^{-s}}{F^{-s+1}}; D) = H^*(\overline{B}^{-s}; \partial),$$

where ∂ is the differential on the tensor product $\widetilde{A} \otimes \widetilde{A} \otimes \cdots \otimes \widetilde{A}$, i.e., $E_1^{-s,*} = \overline{B}^{-s}(\widetilde{H}(A))$ by the Künneth theorem. By the map

$$\theta: \quad \overline{B}^{-s}(\widetilde{H}(A)) \to H(\overline{B}^{-s}(A))$$

in the Künneth theorem, the product induced by the shuffle product of $\overline{B}^{\bullet}$ in E_1 coincides with natural shuffle product defined on $\overline{B}^{\bullet}(\widetilde{H}(A))$.

Using the Künneth theorem, we also can prove that the coproduct induced by the coproduct of $\overline{B}^{\bullet}$ in E_1 coincides with natural coproduct defined on $\overline{B}^{\bullet}(\widetilde{H}(A))$. Hence the natural Hopf algebra structure on $\mathrm{Tor}_{H(A)}(K,K)$ coincides with the Hopf algebra structure induced by spectral sequence.

Q.E.D.

Theorem 1.5 (Eilenberg-Moore) *Suppose B is a connected pointed topological space with $H_1(B) = 0$. Then there is a natural isomorphism of algebra $\theta^* : \mathrm{Tor}_{C^*(B)}(K,K) \to H^*(\Omega B)$, where $C^*(B)$ is the cochain complex of B.*

Proof See [1] p233 (7.10) and [1] p237 (7.13).

Corollary 1.6 *If the pointed topological space B is simply connected, then there is a spectral sequence with $E_2 = \mathrm{Tor}_{H^*(B)}(K,K)$ converging to $H^*(\Omega B)$ as a Hopf algebra.*

Proof 1.6 is an immediate conclusion of 1.5 and 1.4.

Q.E.D.

If A and A' are differential graded augmented algebras over K, and $h: A \to A'$ is a homomorphism of differential graded augmented K-algebras, then h induces a map

$$\overline{B}(h): \overline{B}^{\bullet}(A,K) \longrightarrow \overline{B}^{\bullet}(A',K).$$

$\overline{B}(h)$ commutes with D and the coproduct Δ. So it induces a homomorphism of Hopf algebras

$$\mathrm{Tor}_h(1,1): \quad \mathrm{Tor}_A(K,K) \longrightarrow \mathrm{Tor}_{A'}(K,K).$$

If we consider the duals of all the above definitions and theorems, etc., we can get a similar result for the homology case.

Let C be a differential graded coaugmented coalgebra over K (see [7] p217) with differential d and coproduct $\Delta : C \to C \otimes C$. Let M be a right C-comodule and N be a left C-comodule. Define the *cotensor product* $M[\ \ ]_C N = \ker[i: M \otimes N \longrightarrow M \otimes C \otimes N]$, where $i = \Delta_M \otimes 1_N + 1_M \otimes \Delta_N$ with Δ_M and Δ_N being the structure morphisms of the comodules.

Put $\underline{C} = C/K$, so that if C is connected,

$$\underline{C}_n = \begin{cases} 0, & n = 0, \\ C_n, & n > 0. \end{cases}$$

Let $B_{-r}(C,K) = C \otimes \underbrace{\underline{C} \otimes \cdots \otimes \underline{C}}_{r}$, where $B_0(C,K) = C$, and $B_{\bullet}(C,K) = \sum_{r \geq 0} B_{-r}(C)$. Denote $\widetilde{B}_{-r} = K[\ \ ]_C B_{-r}(C,K) = \underbrace{\underline{C} \otimes \ldots \otimes \underline{C}}_{r}$, and $\widetilde{B}_{\bullet} = K[\ \ ]_C B_{\bullet}$. The canonical isomorphism $\widetilde{B}_{-r}(C) \otimes \widetilde{B}_{-s}(C) \to \widetilde{B}_{-r-s}(C)$ induces a product by juxtaposition $\mu : \widetilde{B}_{\bullet}(C) \otimes \widetilde{B}_{\bullet}(C) \to \widetilde{B}_{\bullet}(C)$. Define $\delta' : \quad \widetilde{B}_{-r}(C) \to \widetilde{B}_{-r-1}(C)$ by

$$\delta'(a_1 \otimes a_2 \otimes \cdots \otimes a_r) = \sum_{i=1}^{r} a_1 \otimes \cdots \otimes (\Delta a_i) \otimes \cdots \otimes a_r.$$

Let $\widetilde{D}$ be the boundary of $\widetilde{B}_{\bullet}(C)$ defined by

$$\widetilde{D}(a_1 \otimes a_2 \otimes \cdots \otimes a_r) = \sum_{i=1}^{r} a_1 \otimes \cdots \otimes da_i \otimes \cdots \otimes a_r + \delta'(a_1 \otimes a_2 \otimes \cdots \otimes a_r).$$

Also we denote $[\![a_1|a_2|\cdots|a_r]\!] = a_1 \otimes \cdots \otimes a_r$.

Definition 1.1′ (See [2]) $\mathrm{Cotor}^C(K,K) = H(\widetilde{B}_\bullet(C), \widetilde{D})$.

Proposition 1.7 *Let (B,b) be a 1-connected space with base point. There is a natural algebra structure on* $\mathrm{Cotor}^{C_*(B)}(K,K)$*, which is induced by* μ*, and a natural isomorphism* $\mathrm{Cotor}^{C_*(B)}(K,K) \cong H_*(\Omega B)$.

Proof See [2] p410.

1.6′ *If B is 1-connected, then there is a spectral sequence with*

$$E^2 = \mathrm{Cotor}^{H_*(B)}(K,K)$$

converging to $\mathrm{Cotor}^{C_*(B)}(K,K)$.

1.4′ *The spectral sequence in 1.6′ is a spectral sequence of Hopf algebras, converging to its target as Hopf algebra.*

If C and C' are differential graded coalgebras over K and $h: \quad C \longrightarrow C'$ is a homomorphism of such, then h induces a map $\widetilde{B}(h): \quad \widetilde{B}_\bullet(C) \longrightarrow \widetilde{B}_\bullet(C')$ which commutes with $\widetilde{D}$ and μ. So it induces a homomorphism of Hopf algebras

$$\mathrm{Cotor}^h(1,1): \quad \mathrm{Cotor}^{C_*}(K,K) \longrightarrow \mathrm{Cotor}^{C'_*}(K,K).$$

§2. The computation of $H^*(\Omega S^2 MU(n))$; $n \geq 1$.

Let $BU(n)$ be the classifying space for complex n-plane bundles (the limit of complex Grassmann manifolds $lim_{m\to\infty} G_n(\mathbf{C}^m)$), $\gamma^n(\mathbf{C}^\infty)$ be the canonical complex n-plane bundle over $BU(n)$, $E(\gamma^n)$ be the total apace of γ^n and $MU(n)$ be the Thom complex of $E(\gamma^n)$.

Theorem 2.1 *The cohomology $H^*(BU(n);Z)$ is the polynomial ring over Z generated by Chern classes $c_1, c_2 \ldots, c_n$. There are no polynomial relations among these n generators.*
Proof See [4] p161 (14.5).

Theorem 2.2 *One has an exact sequence*

$$0 \longleftarrow H^*(BU(n-1);K) \xleftarrow{\alpha} H^*(BU(n);K) \xleftarrow{\beta} H^*(MU(n);K) \longleftarrow 0$$

where β is identified with zero section, $\beta(u) = c_n$ identifying $H^*(MU(n);K)$ with ideal generated by c_n in $H^*(BU(n);K)$ where u is the Thom class.
Proof See [5] p73.

Proposition 2.3 *Let X be a connected space and Λ be a commutative ring (with unit element) such that $H_q(X;\Lambda)$ is a free Λ-module for each $q \geq 0$. Then we have a natural isomorphism of $H_*(\Omega S(X);\Lambda)$ with the tensor algebra $T(\sum_{q>0} H_q(X;\Lambda))$.*
Proof See [3], p22-07 Corollary 2.

Lemma 2.4

$$H^*(\Omega S^2 MU(n)) \cong \overline{B}^\bullet(H(S^2 MU(n)))$$

as vector space.
Proof By Proposition 2.3, we have

$$H_*(\Omega S^2 MU(n)) = T(\sum_{q>0} H_q(SMU(n))).$$

Then

$$\begin{aligned} H^*(\Omega S^2 MU(n)) &= (T(\sum_{q>0} H_q(SMU(n))))^* \\ &= \overline{B}^\bullet(H^*(S^2 MU(n))) \end{aligned}$$

as a vector space.

Q.E.D.

We can use the Eilenberg-Moore spectral sequence to compute $H^*(\Omega S^2 MU(n))$. Since the multiplication on $H^*(S^2 MU(n))$ is trivial, the differential on $\overline{B}^\bullet(H^*(S^2 MU(n)))$ is trivial. Thus

$$\mathrm{Tor}_{H^*(S^2 MU(n))}(K,K) = \overline{B}^\bullet(H^*(S^2 MU(n))).$$

By Lemma 2.4, the Eilenberg-Moore spectral sequence

$$\mathrm{Tor}_{H^*(S^2MU(n))}(K,K) \Longrightarrow \mathrm{Tor}_{C^*(S^2MU(n))}(K,K)$$

collapses. Hence

$$\overline{B}^{\bullet}(H^*(S^2MU(n))) = H^*(\Omega S^2MU(n)).$$

Lemma 2.5 *Let A be any algebra. We have $\alpha^2 = 0$ for any $\alpha \in \mathrm{Tor}_A^n(K,K)$, where $n > 0$.*

Proof For any

$$\alpha = [a_1|a_2|\cdots|a_n] \in \overline{B}^{\bullet}(A),$$

we have

$$\alpha^2 = \sum_{(n,n)-\text{shuffles } \sigma} [w_{\sigma^{-1}(1)}|w_{\sigma[B-1(2)}|\cdots|w_{\sigma^{-1}(2n)}].$$

For any (n,n)-shuffle permutation σ, there exists one and only one (n,n)-shuffle permutation σ' such that

$$\sigma'(n+i) = \sigma(i) \qquad i = 1,2,\ldots,n;$$
$$\sigma'(j) = \sigma(n+j) \qquad j = 1,2,\ldots,n.$$

Then

$$[w_{\sigma^{-1}(1)}|w_{\sigma^{-1}(2)}|\cdots|w_{\sigma^{-1}(2n)}]$$
$$=[w_{\sigma'^{-1}(1)}|w_{\sigma'^{-1}(2)}|\cdots|w_{\sigma'-1(2n)}].$$

Since $\sigma \neq \sigma'$, and $K = F_2$, the terms of α^2 cancel out in pairs. Then $\alpha^2 = 0$. Thus for any $\alpha \in \overline{B}^{\bullet}(A)$, we have $\alpha^2 = 0$.

Q.E.D.

Theorem 2.6 (Borel) *If A is a connected Hopf algebra over the perfect field K, the multiplication in A is commutative, and the underlying graded vector space of A is of finite type, then as an algebra, A is isomorphic with a tensor product $\bigotimes_{i\in I} A_i$ of Hopf algebras A_i, where A_i is a Hopf algebra with a single generator x_i.*

Proof See [7] p255 (7.11).

Lemma 2.7 $\mathrm{Tor}_{H^*(S^2MU(n))}(K,K)$ *is an exterior algebra,*

Proof By Theorem 2.6,

$$\mathrm{Tor}_{H^*(S^2MU(n))}(K,K) = \bigotimes_{i\in I} A_i.$$

By Lemma 2.5, we have $x_i^2 = 0$ where x_i is the generator of A_i for $i \in I$.

Q.E.D.

Lemma 2.8 *We have $\alpha^2 = 0$, for all $\alpha \in H^*(\Omega S^2 MU(n))$ with $\deg\alpha > 0$.*

Proof Let $H = H^*(\Omega S^2MU(n))$, a connected filtered Hopf algebra whose associated graded Hopf algebra $\mathrm{Gr}H$ is E_∞. The spectral sequence collapses by Lemma 2.4,

$$\mathrm{Gr}H = E_2 = \overline{B}^{\bullet}(H^*(S^2MU(n))).$$

But the primitives in $\overline{B}^{\bullet}(H^*(S^2MU(n)))$ are exactly $\overline{B}^{-1}(H^*(S^2MU(n)))$, which is entirely in odd degrees. Thus all nonzero primitives in $\mathrm{Gr}H$ have odd degree.

Let α be any primitive in H, with filtration exactly $-k$. Then α gives a nonzero element $\overline{\alpha} \in \mathrm{Gr}H$ which is also primitive. Then

$$\deg\alpha = \deg\overline{\alpha}$$

must be odd. Hence we have that all nonzero primitives in H have odd degree.

We use induction on degree to prove the result.

Suppose $\alpha^2 = 0$ for any $\alpha \in H^s$ and $0 < s < n$, which is vacuously true for $n = 1$. For $\alpha \in H^n$, we have

$$\Delta\alpha^2 = 1\otimes\alpha^2 + \alpha^2\otimes 1 + \sum_i \alpha_i'^2 \otimes \alpha_i''^2,$$

where

$$\Delta\alpha = \alpha\otimes 1 + 1\otimes\alpha + \sum \alpha_i' \otimes \alpha_i''$$

and $\alpha_i' \in H^s$, $\alpha_i'' \in H^r$ with $s + r = n$, $s > 0$, $r > 0$. Then by the inductive hypothesis

$$\Delta\alpha^2 = 1\otimes\alpha^2 + \alpha^2\otimes 1.$$

So α^2 is primitive, has even degree, and must therefore be zero.

Q.E.D.

Theorem 2.9 *$H^*(\Omega S^2 MU(n))$ is an exterior algebra.*

Proof Since $H^*(\Omega S^2 MU(n))$ is a Hopf algebra and $\alpha^2 = 0$ for any

$$\alpha \in H^*(\Omega S^2 MU(n)),$$

Theorem 2.9 follows from Theorem 2.6 immediately.

Q.E.D.

§3. The cohomology of the loop space over $MU(n), n \geq 2$.

Let C_s $(\subset \overline{B}^\bullet(H^*(MU(n)))$ be the vector space spanned by

$$\{[c_n^{k_1^n} \cdots c_2^{k_1^2} c_1^{k_1^1} | \cdots | c_n^{k_m^n} \cdots c_2^{k_m^2} c_1^{k_m^1}] : k_1^n + \ldots k_m^n = s\},$$

$$C_0 = K$$

and $\widehat{C}_1$ be the vector space spanned by

$$\{[c_n c_{n-1}^{k^{n-1}} \cdots c_2^{k^2} c_1^{k^1}] : k^1 + \ldots + k^{n-1} > 0\}.$$

(C_s, d_1) is a subcomplex of $(\overline{B}^\bullet(H^*(MU(n))), d_1)$ and

$$\overline{B}^\bullet(H^*(MU(n)) = \bigoplus_{s=0}^{\infty} C_s,$$

where $d_1 = \delta^*$ in §1. Here it is convenient to write $H_s(\quad)$ for the homology of the bar construction.

Theorem 3.1 *For $s \geq 1$,*

$$H_s(C_s) \cong C_1 \otimes \underbrace{\widehat{C}_1 \otimes \ldots \otimes \widehat{C}_1}_{s-1},$$

and $H_s(C_s)$ is a subgroup of $\overline{B}^{-s} \cap C_s$. $H_t(C_s) = 0$ for $t \neq s$.

Proof For $s = 1$, we have $H_1(C_1) = C_1$. The result holds. Suppose that for $s-1$ the result holds. For s, let C_s' be the subcomplex of C_s spanned by

$$\{[c_n^{k_1^n} \ldots c_1^{k_1^1} | \cdots | c_n^{k_m^n} \ldots c_1^{k_m^1}] : k_1^n + \ldots k_m^n = s, \quad \text{and} \quad k_m^n > 1\}$$

and h be the chain map

$$h: \ C_{s-1} \longrightarrow C'_s$$

given by

$$h[c_n^{k_1^n} \dots c_1^{k_1^1} | \cdots | c_n^{k_m^n} \dots c_1^{k_m^1}]$$
$$= [c_n^{k_1^n} \dots c_1^{k_1^1} | \cdots | c_n^{k_{m-1}^n} \dots c_1^{k_{m-1}^1} | c_n^{k_m^n+1} \dots c_1^{k_m^1}].$$

h is an isomorphism on the chain level. The short exact sequence of complexes

$$0 \to C'_s \xrightarrow{\alpha} C_s \xrightarrow{\beta} \frac{C_s}{C'_s} \longrightarrow 0$$

induces a long exact sequence

$$\cdots \longrightarrow H_{t+1}(\frac{C_s}{C'_s}) \longrightarrow H_t(C'_s) \longrightarrow H_t(C_s) \longrightarrow H_t(\frac{C_s}{C'_s}) \longrightarrow H_{t-1}(C'_s) \longrightarrow \cdots.$$

The theorem follows from the following two lemmas.

Lemma 3.2

$$H_s(\frac{C_s}{C'_s}) \cong H_{s-1}(C_{s-1}) \otimes C_1.$$

$$H_t(\frac{C_s}{C'_s}) = 0 \quad for \quad t \neq s.$$

Proof Define the chain map

$$g: \ C_{s-1} \otimes C_1 \to \frac{C_s}{C'_s}$$

by

$$g[c_n^{k_1^n} \dots c_1^{k_1^1} | c_n^{k_2^n} \dots c_1^{k_2^1} | \cdots | c_n^{k_m^n} \dots c_1^{k_m^1}] \otimes [c_n c_{n-1}^{k^{n-1}} \dots c_1^{k^1}]$$
$$= [c_n^{k_1^n} \dots c_1^{k_1^1} | c_n^{k_2^n} \dots c_1^{k_2^1} | \cdots | c_n^{k_m^n} \dots c_1^{k_m^1} | c_n c_{n-1}^{k^{n-1}} \dots c_1^{k^1}].$$

Since g is an isomorphism on the chain level,

$$H(\frac{C_s}{C'_s}) \cong H(C_{s-1} \otimes C_1).$$

By the Künneth theorem, we have

$$
\begin{aligned}
H_t(C_{s-1} \otimes C_1) &= H_{t-1}(C_{s-1}) \otimes H_1(C_1) \\
&= H_{t-1}(C_{s-1}) \otimes C_1.
\end{aligned}
$$

Thus the lemma holds.

Lemma 3.3 *The map*

$$
H_s(\frac{C_s}{C'_s}) \xrightarrow{\partial_*} H_{s-1}(C'_s)
$$

is surjective and takes $z \otimes [c_n]$ *to* z, *where*

$$
H_{s-1}(C_{s-1}) \otimes C_1 \cong H_s(\frac{C_s}{C'_s})
$$

and

$$
H_{s-1}(C'_s) \cong H_{s-1}(C_{s-1}).
$$

Proof Recall that

$$
\partial_*\{b\} = \{\alpha_*^{-1} d_1 \beta_*^{-1} b\}.
$$

There is an obvious lifting

$$
\widetilde{g}: \ C_{s-1} \otimes C_1 \longrightarrow C_s
$$

of g, by

$$
\widetilde{g}([x_1|\cdots|x_{s-1}] \otimes y) = [x_1|\cdots|x_{s-1}|y],
$$

which is not a chain map. Instead, from the definition of d_1,

$$
d_1\widetilde{g}(a \otimes [c_n]) = \widetilde{g}(d_1 a \otimes [c_n]) + h(a),
$$

for

$$
a = [c_n c_{n-1}^{k_1^{n-1}} \dots c_1^{k_1^1} | c_n c_{n-1}^{k_2^{n-1}} \dots c_1^{k_2^1} | \cdots | c_n c_{n-1}^{k_{s-1}^{n-1}} \dots c_1^{k_{s-1}^1}]
$$

and therefore for any $a \in C_{s-1}$ in filtration index $s-1$. If a is a cycle representing z, this gives $\partial_* g_*(z \otimes [c_n]) = z$.

Proof of Remainder of Theorem 3.1

From the long exact sequence and Lemma 3.3,

$$\begin{aligned} H_s(C_s) &\cong Ker[\partial_* :\ H_s(\frac{C_s}{C'_s}) \longrightarrow H_{s-1}(C'_s)] \\ &\cong Ker[\partial_* :\ H_{s-1}(C_{s-1}) \otimes C_1 \longrightarrow H_{s-1}(C_{s-1})] \\ &\cong H_{s-1}(C_{s-1}) \otimes \widehat{C}_1 \\ &\cong C_1 \otimes \widehat{C}_1 \otimes \cdots \otimes \widehat{C}_1 \otimes \widehat{C}_1. \end{aligned}$$

Q.E.D.

Corollary 3.4 *Every element of* $\mathrm{Tor}_{H^*(MU(n))}(K, K)$ *is represented by a cycle in* $\sum_{s=0}^{\infty} \overline{B}^{-s} \cap C_s$.

Proof Since

$$\begin{aligned} \mathrm{Tor}_{H^*(MU(n))}(K, K) &= H(\overline{B}^{\bullet}(H^*(MU(n)))) \\ &= H(\sum_{s=0}^{\infty} C_s) \\ &= \sum_{s=0}^{\infty} H(C_s), \end{aligned}$$

the result follows from Theorem 3.1 at once.

Q.E.D.

Let CP^{∞} be the complex projective space. Since $BU(1) = CP^{\infty}$, $MU(1) \cong BU(1)$ and $S^2 \subset CP^{\infty}$, let

$$f :\quad S^2MU(n-1) \longrightarrow MU(2)$$

be the map $g \circ I$, where

$$I :\quad S^2MU(n-1) = S^2 \wedge MU(n-1) \longrightarrow CP^{\infty} \wedge MU(n-1)$$

is the inclusion and

$$g :\quad CP^{\infty} \wedge MU(n-1) \longrightarrow MU(n)$$

is induced by the Whitney sum. f induces a cohomology homomorphism

$$f^* :\ H^*(MU(n)) \to H^*(S^2MU(n-1))$$

with

$$f^*(c_n^{k^n} c_{n-1}^{k^{n-1}} \dots c_1^{k^1}) = \begin{cases} 0, & k^n > 1, \\ i \otimes c_{n-1}^{k^{n-1}+1} \dots c_1^{k^1}, & k^n = 1, \end{cases}$$

where i is the generator of $H^2(S^2)$.

Proposition 3.5 *The map*

$$f^* : \quad H^*(MU(n)) \longrightarrow H^*(S^2 MU(n-1))$$

induces an injective map

$$\mathrm{Tor}_{H^*(MU(n))}(K,K) \longrightarrow \mathrm{Tor}_{H^*(S^2 MU(n-1))}(K,K).$$

Proof Since $\sum_{s=0}^{\infty} \overline{B}^{-s} \cap C_s$ is spanned by

$$\{[c_n c_{n-1}^{k_1^{n-1}} \dots c_1^{k_1^1} | c_n c_{n-1}^{k_2^{n-1}} \dots c_1^{k_2^1} | \cdots | c_n c_{n-1}^{k_m^{n-1}} \dots c_1^{k_m^1}] : \quad k_i^j \geq 0\},$$

the map

$$\overline{B}(f^*) : \sum_{s=1}^{\infty} \overline{B}^{-s} \cap C_s \longrightarrow \overline{B}^{\bullet}(H^*(S^2 MU(n-1)))$$

is injective. Also (see §2)

$$\mathrm{Tor}_{H^*(S^2 MU(n-1))}(K,K) = \overline{B}^{\bullet} H^*(S^2 MU(n-1)).$$

Hence by 3.4

$$\mathrm{Tor}_{H^*(MU(n))}(K,K) \longrightarrow \mathrm{Tor}_{H^*(S^2 MU(n-1))}(K,K)$$

is injective.

Q.E.D.

Theorem 3.6 *The spectral sequence*

$$\mathrm{Tor}_{H^*(MU(n))}(K,K) \Longrightarrow \mathrm{Tor}_{C^*(MU(n))}(K,K)$$

collapses.

Proof We denote the spectral sequence

$$\mathrm{Tor}_{H^*(S^2MU(n-1))}(K,K) \Longrightarrow \mathrm{Tor}_{C^*(S^2MU(n-1))}(K,K)$$

as (E', d') with

$$E'_2 = \mathrm{Tor}_{H^*(S^2MU(n-1))}(K,K)$$

and

$$\mathrm{Tor}_{H^*(MU(n))}(K,K) \Longrightarrow \mathrm{Tor}_{C^*(MU(n))}(K,K)$$

as (E, d) with

$$E_2 = \mathrm{Tor}_{H^*(MU(n))}(K,K).$$

Since the spectral sequence (E', d') collapses by Theorem 2.6, $d'_k = 0$ for $k = 1, 2, \cdots$. Since

$$f^* : E_2 \longrightarrow E'_2$$

is injective, $d_2 = 0$. Then $E_3 = E_2$ and

$$f^* : E_3 \longrightarrow E'_3$$

is injective and $d_3 = 0$. Inductively, we obtain

$$d_k = 0 \qquad \text{for} \quad k > 2.$$

Thus the spectral sequence (E, d) collapses.

Q.E.D.

Theorem 3.7 *$H^*(\Omega MU(n))$ is an exterior algebra.*

Proof Since

$$(\Omega f)^* : H^*(\Omega MU(n)) \longrightarrow H^*(\Omega S^2 MU(n-1))$$

is an injective map of Hopf algebras, and $H^*(\Omega S^2 MU(n-1))$ is an exterior algebra, $H^*(\Omega MU(n))$ is exterior algebra, by Theorem 2.6.

Q.E.D.

We would like to find a set of exterior generators. To do this we have to dualize.

§4. The homology of the loop space of $MU(n), n \geq 2$.

Let $b_i \in H_*(CP^\infty)$ be the dual of $c_1^i \in H^*(CP^\infty)$. It is known that the Whitney sum

$$CP^\infty \wedge CP^\infty \longrightarrow MU(2)$$

induces a surjective homomorphism on homology

$$H_*(CP^\infty \wedge CP^\infty) \longrightarrow H_*(MU(2))$$

by

$$b_i \otimes b_j \mapsto \begin{cases} b_i \circ b_j & \text{if } i \leq j \\ b_j \circ b_i & \text{if } i > j, \end{cases}$$

and that $\widetilde{H}_*(MU(2))$ has a basis

$$\{b_i \circ b_j : 0 < i \leq j\}.$$

The notation $\circ$ is from [9]. Inductively the Whitney sum

$$CP^\infty \wedge MU(n-1) \longrightarrow MU(n)$$

gives a surjective homomorphism on homology

$$H_*(CP^\infty \wedge MU(n-1)) \longrightarrow H_*(MU(n))$$

by

$$b_i \otimes b_{i_1} \circ b_{i_2} \circ \ldots b_{i_{n-1}} \mapsto b_{i_1} \circ \ldots b_{i_j} \circ b_i \circ b_{i_{j+1}} \circ \ldots b_{i_{n-1}},$$

where $i_1 \leq \ldots i_j \leq i \leq i_{j+1} \leq \ldots \leq i_{n-1}$. The basis of $\widetilde{H}_*(MU(n))$ is

$$\{b_{i_1} \circ b_{i_2} \circ \ldots \circ b_{i_n} : 0 < b_{i_1} \leq b_{i_2} \leq \ldots \leq b_{i_n}\}$$

The inclusion map

$$I: \quad S^2MU(n-1) \longrightarrow CP^\infty \wedge MU(n-1)$$

induces an injective homomorphism on homology

$$I_*: \quad H_*(S^2MU(n-1)) \longrightarrow H_*(CP^\infty \wedge MU(n-1))$$

by

$$I_*(i \otimes b_{i_1} \circ b_{i_2} \circ \ldots \circ b_{i_{n-1}}) = b_1 \otimes b_{i_1} \circ b_{i_2} \circ \ldots \circ b_{i_{n-1}}.$$

The map

$$f: \quad S^2MU(n-1) \longrightarrow MU(n)$$

in §3 induces a homology homomorphism

$$f_*: \quad H_*(S^2MU(n-1)) \longrightarrow H_*(MU(n))$$

with

$$f_*(i \otimes b_{i_1} \circ \ldots b_{i_{n-1}}) = b_1 \circ b_{i_1} \circ \ldots \circ b_{i_{n-1}},$$

where i is the generator of $H_2(S^2)$.

Put $G = \widetilde{H}_*(SMU(n-1))$, where S is the suspension isomorphism

$$S_p: \quad H_p(SMU(n-1)) \longrightarrow H_{p-1}(MU(n-1))$$

for all $p > 1$. Denote

$$S^{-1}(b_{i_1} \circ b_{i_2} \circ \ldots \circ b_{i_{n-1}}) = g_{i_1, i_2, \ldots i_{n-1}}.$$

G has a basis

$$\{g_{i_1, i_2, \ldots i_{n-1}}: \quad 0 < i_1 \leq i_1 \leq \ldots \leq i_{n-1}\},$$

and

$$g_{i_1, \ldots, i_j, \ldots, i_k, \ldots i_{n-1}} = g_{i_1, \ldots, i_k, \ldots, i_j, \ldots i_{n-1}}.$$

By the definition of $\widetilde{B}^\bullet$, we have

$$\widetilde{B}^\bullet(H_*(S^2MU(n-1))) = TG,$$

the tensor algebra on G.

Let J be the ideal in TG generated by

$$\{\sum_{i_1=1}^{m_1-1} \sum_{i_2=1}^{m_2-1} \cdots \sum_{i_{n-1}=1}^{m_{n-1}-1} g_{i_1, i_2, \ldots, i_{n-1}} \otimes g_{m_1-i_1, m_2-i_2, \ldots, m_{n-1}-i_{n-1}}: \ m_j > 1, j = 1, 2, \ldots n-1.\}.$$

Theorem 4.1 $\mathrm{Cotor}^{f_*}(1,1)$ *is surjective and the kernel of* $\mathrm{Cotor}^{f_*}(1,1)$ *contains* J.

Proof Since

$$f: \quad S^2MU(n-1) \longrightarrow MU(n)$$

induces an injective homomorphism

$$\mathrm{Tor}_{f^*}(1,1): \ \mathrm{Tor}_{H^*(MU(n))}(K,K) \longrightarrow \mathrm{Tor}_{H^*(S^2MU(n-1))}(K,K),$$

f induces a surjective homomorphism

$$\mathrm{Cotor}^{f_*}(1,1): \ \mathrm{Cotor}^{H_*(S^2MU(n-1))}(K,K) \to \mathrm{Cotor}^{H_*(MU(n))}(K,K).$$

Since

$$\mathrm{Cotor}^{H_*(S^2MU(n-1))}(K,K) = \widetilde{B}^\bullet(H_*(S^2MU(n-1))),$$

$$\mathrm{Cotor}^{H_*(MU(n))}(K,K) \widetilde{=} \frac{\widetilde{B}^\bullet(H_*(S^2MU(n-1)))}{\mathrm{Ker}\,\mathrm{Cotor}^{f^*}(1,1)}.$$

Since

$$\widetilde{B}(f_*): \quad \widetilde{B}^\bullet(H_*(S^2MU(n-1))) \longrightarrow \widetilde{B}^\bullet(H_*(MU(n)))$$

is injective, it induces an isomorphism,

$$\mathrm{Ker}\,\mathrm{Cotor}^{f_*}(1,1) \widetilde{=} \mathrm{Im}\,\delta' \cap \mathrm{Im}\,\widetilde{B}(f_*),$$

where δ' is the differential of the spectral sequence defined on §1.

$$\mathrm{Cotor}^{H_*(MU(n))}(K,K) \Longrightarrow H_*(\Omega MU(n)).$$

In $H_*(CP^\infty))$, we have

$$\Delta b_n = b_n \otimes 1 + b_{n-1} \otimes b_1 + b_{n-2} \otimes b_2 + \ldots + 1 \otimes b_n.$$

The homomorphism I_* and Whitney sum are homomorphisms of coalgebras, so is f_*. In $H_*(MU(n))$, we therefore have

$$\begin{aligned}
&\Delta(b_2 \circ b_{m_1} \circ \ldots \circ b_{m_{n-1}}) \\
=&1 \otimes (b_2 \circ b_{m_1} \circ \ldots \circ b_{m_{n-1}}) + (b_2 \circ b_{m_1} \circ \ldots \circ b_{m_{n-1}}) \otimes 1 \\
+&\sum_{i_1=1,i_2=1,\ldots,i_{n-1}=1}^{m_1-1,m_2-1,\ldots,m_{n-1}-1} (b_1 \circ b_{i_1} \circ \ldots \circ b_{i_{n-1}}) \otimes (b_1 \circ b_{m_1-i_1} \circ \ldots \circ b_{m_{n-1}-i_{n-1}})
\end{aligned}$$

for $n \geq 2$. Since

$$f_*(g_{i_1,i_2,\ldots,i_{n-1}}) = f_*[\![i \otimes b_{i_1} \circ \ldots \circ b_{i_{n-1}}]\!] = [\![b_1 \circ b_{i_1} \circ \ldots \circ b_{i_{n-1}}]\!],$$

we have

$$\widetilde{B}(f_*)(J) \subset \operatorname{Im}\delta',$$

as required. Thus the theorem holds.

Q.E.D.

Lemma 4.2 *TG/J is spanned by*

$$\{g_{i_1^1,i_2^1,\dots i_{n-1}^1} \otimes g_{i_1^2,i_2^2,\dots i_{n-1}^2} \otimes \dots \otimes g_{i_1^l,i_2^l\dots i_{n-1}^l}\},$$

where $i_{n-1} > 1$, for $r > 1$; $0 < i_1^r \le \dots \le i_{n-1}^r$ all $r > 0$

Proof Recall that

$$g_{1,1,\dots,1} \otimes g_{1,1,\dots,1} = 0.$$

For any other $(m_1, m_2, \dots, m_{n-1})$ with $m_{n-1} > 1$ and $0 < m_1 \le m_2 \le \dots \le m_{n-1}$, we have

$$\begin{aligned}
&g_{m_1,m_2,\dots,m_{n-1}} \otimes g_{1,1,\dots,1}\\
=&g_{1,1,\dots,1} \otimes g_{m_1,m_2,\dots,m_{n-1}}\\
+&\Big(\sum_{i_1=1}^{m_1-1} \cdots \sum_{i_{n-1}=1}^{m_{n-1}-1} g_{i_1,i_2,\dots,i_{n-1}} \otimes g_{m_1-i_1,\dots m_{n-1}-i_{n-1}}\\
-&g_{1,\dots,1} \otimes g_{m_1,\dots,m_{n-1}} - g_{m_1,\dots,m_{n-1}} \otimes g_{1,\dots,1}\Big)
\end{aligned}$$

in TG/J. So we can move all $g_{1,1,\dots,1}$'s to the left.

Q.E.D.

Theorem 4.3 *The elements in 4.2 give a basis of TG/J and* $\mathrm{Cotor}^{H_*(MU(n))}(K,K)$.

Proof Comparing Lemma 4.2 and Theorem 3.3, we have that TG/J and $\mathrm{Tor}_{H^*(MU(n))}(K,K)$ have the same size. Thus $\operatorname{Ker}\mathrm{Cotor}^{f_*}(1,1) = J$. That means

$$\frac{TG}{J} = \mathrm{Cotor}^{H_*(MU(n))}(K,K).$$

Q.E.D.

§5. The primitives of $H_*(\Omega MU(n)), n \ge 2$.

Definition 5.1 If $c \in C$, where C is an augmented coalgebra over K and

$$\Delta c = 1 \otimes c + c \otimes 1,$$

then c is called a *primitive* element of C. The set

$$PC = \{c : \quad c \text{ is primitive in } C\}$$

is called the *primitive module* over K of C.

Recall that $K = F_2$.

Definition 5.2 A *restricted Lie algebra* over K is a Lie algebra L together with a function $\xi : L_n \longrightarrow L_{2n}$ satisfying

$$\xi(x+y) = \xi\, x + \xi\, y + [x, y]$$

and

$$[x, \xi\, y] = [[x, y], y].$$

Any algebra A over F_2 can be made into a restricted Lie algebra by setting

$$[x, y] = xy - yx, \quad \xi x = x^2.$$

The axioms hold since

$$(x+y)^2 = x^2 + y^2 + [x, y]$$

and

$$[x, y^2] = [[x, y], y].$$

Proposition 5.3 *PC is a restricted Lie algebra over K.*

Proof We can check the definition directly.

Q.E.D.

Definition 5.4 If L is a restricted Lie algebra over K, the *universal enveloping algebra* of L is an algebra $V(L)$ together with a morphism of restricted Lie algebras $i_L : \ L \to V(L)$

such that if A is an algebra and $f: L \to A$ is a morphism of restricted Lie algebras, there is a unique morphism of algebras $\widetilde{f}: V(L) \to A$ such that the diagram

$$\begin{array}{ccc} L & \xrightarrow{i_L} & V(L) \\ & \underset{f}{\searrow} & \downarrow \widetilde{f} \\ & & A \end{array}$$

is commutative.

The universal enveloping algebra is easily constructed. Put

$$V(L) = T(L)/I$$

where $T(L)$ is the tensor algebra of L and I is the ideal of $T(L)$ generated by all the elements

$$\{x \otimes y + y \otimes x + [x, y] \quad \text{and} \quad x \otimes x + \xi x\}.$$

Proposition 5.5 *If L is a restricted Lie algebra over K with basis*

$$\{x_1, \ x_2, \ x_3, \ \dots \},$$

then $V(L)$ has the basis

$$\{x_{i_1} x_{i_2} \cdots x_{i_k} : \ i_1 < i_2 < \cdots < i_k, \ k \geq 0\}$$

Proof See [7] p247 (6.19).

Proposition 5.6

$$P(H_*(\Omega S^2 MU(n-1))) = L(G),$$

where $L(G)$ is the free restricted Lie algebra on G.

Proof Since G generates

$$H_*(\Omega S^2 MU(n-1)) = TG,$$

and $G \subset PTG$ by definition, TG is primitively generated. By [7] (6.10), we have

$$TG = V(PTG).$$

By definition,

$$V(L(G)) = TG,$$

since both sides have the same universal property. We deduce from 5.5 that

$$PTG = L(G).$$

Q.E.D.

Denote

$$H' = \mathrm{Cotor}^{H_*(MU(n))}(K, K),$$

and

$$\overline{f}_* = \mathrm{Cotor}^{f_*}(1, 1): \ TG \longrightarrow H'.$$

Then

$$H' = \frac{TG}{J}$$

is also primitively generated, and So $H' = VPH'$.

Theorem 5.7 *The homomorphism*

$$\overline{f}_*: \ \ TG \longrightarrow H'$$

induces a surjective homomorphism

$$P(\overline{f}_*): \quad PT(G) \longrightarrow PH'.$$

Proof Put $M = \mathrm{Im}(P\overline{f}_*)$. Since $PT(G)$ and PH' are restricted Lie algebras and $\overline{f}_*$ is a homomorphism of restricted Lie algebras, M is a restricted Lie algebra. Since $TG = VPTG, \quad H' = VPH'$ by [7] (6.10), the homomorphism

$$\begin{array}{ccccc} VPTG & & \longrightarrow & & VPH \\ & \searrow & & \nearrow & \\ & & VM & & \end{array}$$

is an epimorphism. So the injective homomorphism

$$VM \longrightarrow VPH'$$

is an epimorphism. Thus $M = PH'$ by 5.5.

Q.E.D.

Note that the kernel ideal J is generated by $J \cap PTG = J \cap LG$, because the generators can be written

$$\sum_{i_1=1}^{\frac{m_1+1}{2}} \sum_{i_2=1}^{\frac{m_2+1}{2}} \cdots \sum_{i_{n-1}=1}^{\frac{m_{n-1}+1}{2}} \sum_{\mathcal{A}} [g_{i_1,i'_2,\ldots i'_{n-1}}, g_{m_1+1-i_1,i''_2,\ldots,i''_{n-1}}] + g^2_{\frac{m_1+1}{2},\frac{m_2+1}{2},\ldots,\frac{m_{n-1}+1}{2}}$$

if all $m_1, m_2, \ldots, m_{n-1}$ are odd,

$$\sum_{i_1=1}^{[\frac{m_1+1}{2}]} \sum_{i_2=1}^{[\frac{m_2+1}{2}]} \cdots \sum_{i_{n-1}=1}^{[\frac{m_{n-1}+1}{2}]} \sum_{\mathcal{A}} [g_{i_1,i'_2,\ldots i'_{n-1}}, g_{m_1+1-i_1,i''_2,\ldots,i''_{n-1}}] \qquad (*)$$

otherwise,

where $\sum_{\mathcal{A}}$ is the sum over all such indices

$$\mathcal{A} = \{(i_1, i'_2, \ldots, i'_{n-1};\ m_1 + 1 - i_1, i''_2, \ldots, i''_{n-1})\}$$

that satisfy: either $i'_j = i_j$ and $i''_j = m_j + 1 - i_j$ or $i'_j = m_j + 1 - i_j$ and $i''_j = i_j$. It is noticed that if $i_j = m_j + 1 - i_j$, then only

$$(i_1, \ldots, i'_{j-1}, i_j, i'_{j+1}, \ldots, i'_{n-1};\ n_1 + 1 - i_1, \ldots, i''_{j-1}, m_j + 1 - i_j, i''_{j+1} \ldots, i''_{n-1})$$

is in $\mathcal{A}$ (This has to happen by general nonsense). So

$$PH' = \frac{LG}{J \cap LG},$$

the quotient restricted Lie algebra. We need to find PH'.

Let E be the restricted Lie subalgebra of H' generated by

$$\{g_{i_1,i_2,\ldots,i_{n-1}} :\ 0 < i_1 \leq \ldots \leq i_{n-1},\ \text{and } i_{n-1} > 1\}.$$

Lemma 5.8 *$[g_{1,1,\ldots 1}, E]$ is contained in E, where $[\quad,\quad]$ is the Lie product in H'.*

Proof We show by induction on n that $[g_{1,1,\ldots,1}, x]$ is in E for every basic product x of weight n in the generators $g_{m_1,m_2,\ldots,m_{n-1}}$ of E.

If x has weight 1, then $x = g_{m_1,m_2,\ldots,m_{n-1}}$ with $0 < m_1 \leq \ldots \leq m_{n-1}$ and $m_{n-1} > 1$. Since by $(*)$

$$[g_{1,\ldots,1}, g_{m_1,\ldots,m_{n-1}}] = \begin{cases} \sum_{i_1=1}^{\frac{m_1+1}{2}} \cdots \sum_{i_{n-1}=1}^{\frac{m_{n-1}+1}{2}} \sum_{\mathcal{B}} [g_{i_1,i'_2,\ldots,i'_{n-1}}, g_{m_1+1-i_1,i''_2,\ldots,i''_{n-1}}] & +g^2_{\frac{m_1+1}{2},\ldots,\frac{m_{n-1}+1}{2}} \\ & m_1,\ldots,m_{n-1} \text{ are odd,} \\ \sum_{i_1=1}^{[\frac{m_1+1}{2}]} \cdots \sum_{i_{n-1}=1}^{[\frac{m_{n-1}+1}{2}]} \sum_{\mathcal{B}} [g_{i_1,i'_2,\ldots,i'_{n-1}}, g_{m_1+1-i_1,i''_2,\ldots,i''_{n-1}}] & \text{otherwise,} \end{cases}$$

where $\sum_{\mathcal{B}}$ is the sum over indeices $\mathcal{B}$ and

$$\mathcal{B} = \mathcal{A} - \{(1,1,\ldots,1; m_1, m_2, \ldots, m_{n-1})\},$$

$$[g_{1,1,\ldots,1}, g_{m_1,m_2,\ldots,m_{n-1}}] \in E.$$

Suppose $[g_{1,1,\ldots,1}, x] \in E$ for all x of weight $< n$. Given $x \in E$ of weight n, we have $x = [z, y]$ with weight $(z) < n$ and weight $(y) < n$ or $x = y^2$ with weight $(y) < n$. Since in the first case

$$[g_{1,1,\ldots,1}[z,y]] = [z, [g_{1,1,\ldots,1}, y]] + [y, [g_{1,1,\ldots,1}, z]],$$

and $y \in E$ and $z \in E$,

$$[g_{1,1,\ldots,1}, x] \in E.$$

If $x = y^2$ with weight $(y) < n$, since $y \in E$ and

$$[g_{1,1,\ldots,1}, y^2] = [y, [g_{1,1,\ldots,1}, y]],$$

then

$$[g_{1,1,\ldots,1}, x] \in E.$$

Hence

$$[g_{1,1,\ldots,1}, E] \subset E.$$

Q.E.D.

Corollary 5.9 *PH' is spanned as a vector space by E and $g_{1,1,\ldots,1}$.*

Proof We have $g^2_{1,1,\ldots,1} = 0$.

Q.E.D.

Let $\mathcal{P}(A)$ be the Poincaré series of the module A.

Corollary 5.10 $\mathcal{P}(H') = (1 + t^{2n-1})\mathcal{P}(VE)$.

Proof From 5.5, since $g_{1,1,\ldots,1}$ has degree $2n-1$ and $H' = VPH'$.

Q.E.D.

Let G' be the vector space with basis

$$\{g_{i_1,\ldots,i_{n-1}} : \ 0 < i_1 \leq \ldots \leq i_{n-1}, \ i_{n-1} > 1\}$$

Let LG' be the free restricted Lie algebra on G'. Then there are epimorphisms

$$LG' \longrightarrow E \quad \text{and} \quad VLG' \longrightarrow VE.$$

Proposition 5.11 *E is the free restricted Lie algebra on G'.*

Proof Since by 4.3, every element of TG/J can be written uniquely as $y + g_{1,\ldots,1}z$ with $y, \ z \in TG'$, we have

$$\begin{aligned} \mathcal{P}(H') &= \mathcal{P}(TG/J) \\ &= (1 + t^{2n-1})\mathcal{P}(TG') \\ &= (1 + t^{2n-1})\mathcal{P}(VLG'). \end{aligned}$$

Since by Corollary 5.10

$$\mathcal{P}(H') = (1 + t^{2n-1})\mathcal{P}(VE),$$

we have

$$\mathcal{P}(VE) = \mathcal{P}(VLG').$$

Thus by Proposition 5.5 the epimorphisms above are isomorphisms. E is free on G'.

Q.E.D.

We next describe the structure of the restricted Lie algebra PH'. Since $E = LG'$ is free, define a derivation of restricted Lie algebras

$$d : \quad E \longrightarrow E$$

by

$$dg_{m_1,\ldots,m_{n-1}}$$
$$= \begin{cases} \sum_{i_1=1}^{\frac{m_1+1}{2}} \cdots \sum_{i_{n-1}=1}^{\frac{m_{n-1}+1}{2}} \sum_{\mathcal{B}} [g_{i_1,i'_2,\ldots,i'_{n-1}}, g_{m_1+1-i_1,i''_2,\ldots,i''_{n-1}}] & +g^2_{\frac{m_1+1}{2},\ldots,\frac{m_{n-1}+1}{2}} \\ & m_1,\ldots,m_{n-1} \text{ are odd} \\ \sum_{i_1=1}^{[\frac{m_1+1}{2}]} \cdots \sum_{i_{n-1}=1}^{[\frac{m_{n-1}+1}{2}]} \sum_{\mathcal{B}} [g_{i_1,i'_2,\ldots \subset'_{n-1}}, g_{m_1+1-i_1,i''_2,\ldots,i''_{n-1}}] & \text{otherwise.} \end{cases}$$

and extend by linearity,

$$d[x,y] = [dx,y] + [x,dy],$$

and

$$d(x^2) = [dx,x].$$

This works because E is free.

Then dd is again a derivation, and by working in $VLG' = TG'$, one can verify directly that $dd\, g_{i_1,i_2,\ldots,i_{n-1}} = 0$, so that $dd = 0$. Define

$$[g_{1,1,\ldots,1}, x] = dx \qquad \text{for} \quad x \in E.$$

Then $[g_{1,\ldots,1}, [g_{1,\ldots,1}, x]] = dd\, x = 0$ as required.

So we have

Theorem 5.12 *PH' is spanned by $LG' \cup \{g_{1,\ldots,1}\}$ as a vector space. The Lie product is defined by the structure of LG' and*

$$[g_{1,\ldots,1}, y] = dy \qquad (y \in LG')$$

and $g^2_{1,\ldots,1} = 0$.

Corollary 5.13 *To obtain a set of generators of the exterior algebra $H^*(\Omega MU(n))$, we may take any set of elements that is dual to a basis of PH'.*

§6. <u>The homology of the double loop space of $MU(n), n \geq 2$.</u>

Proposition 6.1 *If A is a Hopf algebra with basis $\{1, a\}$ and*

$$\Delta a = 1 \otimes a + a \otimes 1,$$

then $\operatorname{Cotor}^A(K, K)$ *is a polynomial algebra generated by* $[\![a]\!]$.

Proof By the definition of Cotor , $\widetilde{B}^\bullet$ is a polynomial algebra generated by $[\![a]\!]$. Since the element in a is primitive, $d_1 = 0$. Thus $\operatorname{Cotor}^A(K, K)$ is a polynomial algebra generated by $[\![a]\!]$.

Q.E.D.

Proposition 6.2 *If A and C are coalgebras over K, then*

$$\operatorname{Cotor}^A(K, K) \otimes \operatorname{Cotor}^C(K, K) = \operatorname{Cotor}^{A \otimes C}(K, K)$$

as an algebra.

Proof $B(A) \otimes B(C)$ is an injective resolution of K by $A \otimes C$-comodules. The Künneth theorem applies.

Q.E.D.

If X is a vector space, denote by EX the exterior algebra on X, made into a Hopf algebra with X primitive.

Proposition 6.3 $\operatorname{Cotor}^{EX}(K, K) = K[\,[\![x_1]\!], [\![x_2]\!] \ldots]$, *a polynomial ring, where* $\{x_1,\ x_2,\ \ldots\}$ *is a basis of X.*

Proof From 6.1, 6.2 and direct limits.

Q.E.D.

Let H be any primitively generated Hopf algebra, and let $\{e_1,\ e_2,\ \ldots\}$ be an ordered basis of PH. Define the additive homomorphism

$$h:\ EPH \longrightarrow H$$

by

$$h(e_{i_1} e_{i_2} \ldots e_{i_n}) = e_{i_1} e_{i_2} \ldots e_{i_n},$$

where $i_1 < i_2 < \ldots < i_n$. BThis formula is not valid if the e_i are out of order.

Lemma 6.4 *If H is a Hopf algebra and*

$$x = x_1 x_2 x_3 \cdots x_n \in H$$

where $x_1, \ x_2, \ \cdots x_n$ are primitive in H, then

$$\Delta x = \sum_{i=0}^{n} \sum_{(i,n-i)-\text{shuffle } \sigma} x_{\sigma(1)} \cdots x_{\sigma(i)} \otimes x_{\sigma(i+1)} \cdots x_{\sigma(n)}.$$

Proof Since $\Delta x_1 = 1 \otimes x_1 + x_1 \otimes 1$, the result holds for $n = 1$. Suppose that the result holds for $n-1$. For $x = x_1 x_2 \cdots x_n$, write

$$z = x_1 x_2 \cdots x_{n-1}.$$

Then

$$\Delta z = \sum_i z_i' \otimes z_i''.$$

By the definition of Hopf algebra,

$$\begin{aligned} \Delta x &= (\Delta z)(\Delta x_n) \\ &= \sum_i z_i' x_n \otimes z_i'' + \sum_i z_i' \otimes z_i'' x_n. \end{aligned}$$

These are all the shuffles of $(x_1, \ x_2, \ \ldots x_n)$.

Q.E.D.

Lemma 6.5 *The homomorphism $h: \ EPH \longrightarrow H$ defined above preserves the comultiplication and is an isomorphism of coalgebras.*

Proof By Proposition 5.5, h is an isomorphism. Since the e_i are primitive in EPH as well as in H, the result follows from Lemma 6.4 immediately.

Q.E.D.

Theorem 6.6 *For a primitively generated Hopf algebra H,*

$$\text{Cotor}^H(K, K) = K[[\![e_1]\!], \ [\![e_2]\!], \ldots],$$

where $\{e_1, \, e_2, \, e_3, \, \ldots\}$ is an ordered basis of PH.

Proof Since by 6.5

$$EPH \cong H$$

as a coalgebra and the definition of Cotor only uses the coproduct of H, the result follows from 6.3.

Q.E.D.

Theorem 6.7 *The spectral sequence*

$$\mathrm{Cotor}^{H_*(\Omega S^2 MU(n-1))}(K,K) \Longrightarrow \mathrm{Cotor}^{C_*(\Omega S^2 MU(n-1))}(K,K)$$

collapses.

Proof See [8], p227 Lemma 3.8.

Lemma 6.8 *The spectral sequence*

$$\mathrm{Cotor}^{H_*(\Omega MU(n))}(K,K) \Longrightarrow \mathrm{Cotor}^{C_*(\Omega MU(n))}(K,K)$$

collapses.

Proof Since

$$LG \longrightarrow PH'$$

is surjective, the morphism of polynomial rings

$$K([\![e_1]\!],\ [\![e_2]\!],\cdots) \longrightarrow K([\![e'_1]\!],\ [\![e'_2]\!],\cdots)$$

is surjective, i.e. on E^2-terms

$$\mathrm{Cotor}^{H_*(\Omega S^2 MU(n-1))}(K,K) \longrightarrow \mathrm{Cotor}^{H_*(\Omega MU(n))}(K,K)$$

is surjective. Then the Lemma follows from 6.7.

Q.E.D.

Theorem 6.9 *$H_*(\Omega^2 MU(n))$ is a polynomial algebra. $QH_*(\Omega^2 MU(n))$ has a basis isomorphic to*

$$\{[\![e'_1]\!],\ [\![e'_2]\!],\ \cdots\},$$

where $\{e'_1, e'_2, \ldots\}$ is a basis of PH' and $\deg [\![e'_i]\!] = \deg e'_i - 1$.

Proof Since the spectral sequence collapses by 6.8, so that

$$E^{\infty} = \mathrm{Cotor}^{H'}(K, K)$$

is a polynomial algebra, lifting each generator $[\![e'_i]\!]$ to

$$e''_i \in H_*(\Omega^2 MU(n))$$

arbitrarily, we have that $H_*(\Omega^2 MU(n))$ is a polynomial algebra generated by $e''_i, \quad i = 1, 2, \ldots$.

Q.E.D.

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