REAL JOHNSON-WILSON THEORIES AND NON-IMMERSIONS OF PROJECTIVE SPACES

by

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Abstract

In this paper we compute the second real Johnson-Wilson cohomology $ER(2)$ of the odd-dimensional real projective space RP^{16K+9}. This enables us to solve certain non-immersion problems of projective spaces using obstructions in $ER(2)$-cohomology.

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Contents

1 Introduction

Atiyah [2] developed K-theory with Reality KR, which is in a sense a mixture of the K-theory of real vector bundles KO, the K-theory of self-conjugate bundles KSC, and the K-theory of G-vector bundles over G-spaces K_G. A *Real* vector bundle over a space X with involution$(x \mapsto \overline{x})$ is a complex vector bundle E over X which also has an involution such that the projection $E \rightarrow X$ commutes with the involution and the map of fibers $E_x \rightarrow E_{\overline{x}}$ is anti-linear.

A special class of $\mathbb{Z}/2$ equivariant cohomology theories are called Real-oriented theories. These are Real-oriented in the sense that any Real bundle is orientable with respect to such a theory. The theory of real complex cobordism was developed by Landweber and Araki ([1],[15]) by taking homotopy fixed points of complex cobordism under complex conjugation. The language of equivariant topology has changed quite substantially ever since then. The equivariant stable homotopy category developed by May-Lewis-Steinberger [14] has provided the basic framework for equivariant topology and cohomology theories for all current work in the field.

The $\mathbb{Z}/2$-equivariant Johnson-Wilson spectrum $E\mathbb{R}(n)$ was first constructed by Hu and Kriz in [9]. Kitchloo and Wilson [12] have used the homotopy fixed point spectrum of this to solve certain non-immersion problems of real projective spaces. The homotopy fixed point spectrum $ER(n)$ is $2^{n+2}(2^n-1)$-periodic compared to the $2(2^n-1)$-periodic $E(n)$. $ER(1)$ is $KO_{(2)}$ and $E(1)$ is $KU_{(2)}$.

Kitchloo and Wilson have demonstrated the existence of a stable cofibration connecting $E(n)$ and $ER(n)$,

$$\Sigma^{\lambda(n)} ER(n) \xrightarrow{x} ER(n) \longrightarrow E(n) \tag{1}$$

This leads to a Bockstein spectral sequence for x-torsion. It is known that $x^{2^{n+1}-1} = 0$ so there can be only $2^{n+1}-1$ differentials. For the case of our interest $n = 2$ there are only 7 differentials.

From [10] we know that if there is an immersion of RP^b to $\mathbb{R}^c$ then there is an axial map

$$RP^b \times RP^{2^L-c-2} \to RP^{2^L-b-2}. \tag{2}$$

For $b = 2n$ and $c = 2k$ Don Davis shows in [4] that there is no such map when $n = m+\alpha(m)-1$ and $k = 2m-\alpha(m)$, where $\alpha(m)$ is the number of ones in the binary expression of m by finding an obstruction to James's map (2) in $E(2)$-cohomology. Kitchloo and Wilson get new non-immersion results by computing obstructions in $ER(2)$-cohomology. In this paper we extend Kitchloo-Wilson's results by computing the $ER(2)$-cohomology of the odd projective space RP^{16K+9}. This will give us newer non-immersion results.

In the first section we introduce the preliminary constructions of equivariant stable homotopy theory from [14] and [6]. We introduce the Tate and Borel spectra associated to a G-equivariant spectrum and introduce the filtrations necessary to construct the Tate and Borel spectral sequences.

In the second section define a Real-oriented spectrum following [9] and construct the Real complex cobordism spectrum $M\mathbb{R}$. This is an E_∞-$\mathbb{Z}/2$-ring spectrum. We use the language of [5] to construct Real versions of various complex oriented spectra. In particular, we construct $E\mathbb{R}(n)$ as the Real-oriented spectrum corresponding to the complex-oriented $E(n)$. Next we compute the Borel and Tate spectral sequences for $E\mathbb{R}(n)$ and reproduce the cofibration (1).

In section 3 we use calculations from section 2 to compute the $E\mathbb{R}(n)$-cohomology of $\mathbb{C}P^\infty$. In section 4 we begin calculations for the Bockstein spectral sequence induced by (1). We focus on the cohomology theory $ER(2)$. Finally we recall essential results from [12] and [13] and embark on the computation of the Bockstein spectral sequence for $ER(2)^*(RP^{16K+9})$.

In section 5 we use results from the previous section to obtain obstructions to the axial map of James (2) for different values of b and c, thus resulting in new non-immersions. The main results are the following.

Theorem 1.1. *A 2-adic basis of* $ER(2)^{8*}(RP^{16K+9},*)$ *is given by the elements*

$$\alpha^k u^j,\ (k \geq 0, 1 \leq j \leq 8K+4)$$

$$v_2^4 \alpha^k u^j,\ (k \geq 1, 1 \leq j \leq 8K+4)$$

$$v_2^4 u^j,\ (4 \leq j \leq 8K+4)$$

$$x\alpha^k i_{16K+9},\ xv_2^4\alpha^k i_{16K+9},\ (k \geq 0)$$

Theorem 1.2. *If* $(m, \alpha(m)) \equiv$ *(6,2) or (1,0) mod 8,*

$RP^{2(m+\alpha(m)-1)}$ *does not immerse in* $\mathbb{R}^{2(2m-\alpha(m))+1}$.

2 G-spectra and G-cohomology theories

We will be working in the equivariant stable homotopy category of Lewis-May-Steinberger. Let us begin by recalling some definitions. Let G be a finite group.

A complete universe U is an infinite dimensional real inner product space with G acting through isometries such that U contains a countably infinite sum of all irreducible representations of G as subspace.

A G-spectrum k_G indexed on U associates a based G-space $k_G(V)$ to each finite dimensional G-subspace $V \subset U$ such that for any two G subspaces V and W of U with $V \subset W$ the structure maps $k_G(V) \to \Omega^{W-V} k_G(W)$ are G-homeomorphisms. Here, $W - V$ denotes the orthogonal complement of V in W. For any V, $S(V)$ denotes the unit sphere in V and S^V denotes the one-point compactification $V \cup \infty$ of V, with ∞ as basepoint. Then for a G-space Y, $\Omega^V Y$ denotes the G-space of all based maps $S^V \to Y$.

Let $U = \oplus(V_i)^\infty$ for a complete set of distinct irreducible representations V_i. Then $RO(G, U)$ is the free abelian group generated by the V_i. We define the $RO(G, U)$-graded homology and cohomology theory associated to any G-spectrum k_G. For any virtual representation $a = V - W$ there are sphere G-spectra $S^a = \Sigma^{-W} S^V$, and we let

$$k_G^a(X) = [X \wedge S^{-a}, k_G]_G$$

and

$$k_a^G(X) = [S^a, X \wedge k_G]_G$$

for any G-spectrum X. For a G-space Y let $k_G^a(Y) = k_G^a(\Sigma^\infty Y)$ and similarly for homology; here Σ^∞ is the functor from G-spaces to G-spectra that is left adjoint to the $0th$ space functor [14].

A map of G-spectra $f : E \to F$ is a stable equivalence when $f_*^H : \pi_n(E^H) \to \pi_n(F^H)$ is an isomorphism for all n and all $H \leq G$.

2.1 The Tate Spectrum

We recall necessary definitions and facts from [6]. The various equivariant spectra associated to any given G-spectrum k_G are displayed in the Tate diagram given below. Let X_+ be the disjoint union of the G-space X with a fixed base point and let EG be a contractible free G-space. Let $\widetilde{EG}$ be the unreduced suspension of EG. Then there is a cofibering

$$EG_+ \to S^0 \to \widetilde{EG} \to \Sigma EG_+. \tag{3}$$

Let $F(EG_+, k_G)$ be the function G-spectrum of maps from EG_+ to k_G with diagonal G action. The projection $EG_+ \to S^0$ induces a map of G-spectra

$$\epsilon : k_G = F(S^0, k_G) \to F(EG_+, k_G).$$

Smashing ϵ with the previous cofibering gives the following map of cofiberings of G-spectra, known as the Tate diagram.

$$\begin{tikzcd}
k_G \wedge EG_+ \arrow[r] \arrow[d, "\simeq"'] & k_G \arrow[r] \arrow[d] & k_G \wedge \widetilde{EG} \arrow[d] \\
F(EG_+, k_G) \wedge EG_+ \arrow[r] & F(EG_+, k_G) \arrow[r] & F(EG_+, k_G) \wedge \widetilde{EG}
\end{tikzcd}$$

Notice that the leftmost vertical arrow is always an equivalence. We define the Borel, geometric and Tate spectra associated to k_G respectively as follows:

$$c(k_G) = F(EG_+, k_G)$$

$$g(k_G) = k_G \wedge \widetilde{EG}$$

$$t(k_G) = F(EG_+, k_G) \wedge \widetilde{EG}$$

$$f(k_G) = k_G \wedge EG_+$$

It follows that, up to equivalence, $t(k_G)$ is the cofiber of the composite map

$$k_G \wedge EG_+ \to k_G \to F(EG_+, k_G).$$

The canonical smash product pairing

$$F(X, Y) \wedge F(X', Y') \to F(X \wedge X', Y \wedge Y')$$

and the equivalences

$$EG_+ \wedge EG_+ \simeq EG_+ \qquad \widetilde{EG} \wedge \widetilde{EG} \simeq \widetilde{EG}$$

give the following proposition.

Proposition 2.1. *If k_G is a ring G-spectrum then $c(k_G)$ and $t(k_G)$ are ring G-spectra and the following part of the Tate diagram is a commutative diagram of ring G-spectra:*

$$\begin{tikzcd}
k_G \arrow[r] \arrow[d] & k_G \wedge \widetilde{EG} \arrow[d] \\
c(k_G) \arrow[r] & t(k_G)
\end{tikzcd}$$

The unit of $t(k_G)$ is the smash product of the unit of $c(k_G)$ and the canonical map $S^0 \to \widetilde{EG}$.

Proposition 2.2.

$$t(k_G)^n(X) \simeq f(k_G)^n(\Sigma^{-1}\widetilde{EG} \wedge X)$$

We may identify $c(k_G) \to t(k_G)$ with

$$F(\Sigma^{-1}(\widetilde{EG}/S^0), k_G \wedge EG_+) \to F(\Sigma^{-1}\widetilde{EG}, k_G \wedge EG_+)$$

These are maps of ring spectra.

This will give a morphism of the spectral sequences.

Proof From the exact triangle (3) we construct a diagram whose rows are exact triangles.

$$\begin{tikzcd}
F(S^0, k_G \wedge EG_+) \arrow[r] \arrow[d, equal] & F(\Sigma^{-1}(\widetilde{EG}/S^0), k_G \wedge EG_+) \arrow[r] \arrow[d, equal] & F(\Sigma^{-1}\widetilde{EG}, k_G \wedge EG_+) \arrow[d, equal] \\
k_G \wedge EG_+ \arrow[r] \arrow[d, "\simeq"] & F(EG_+, k_G \wedge EG_+) \arrow[r] \arrow[d, "\simeq"] & F(\Sigma^{-1}\widetilde{EG}, k_G \wedge EG_+) \arrow[d, "\simeq"] \\
F(EG_+, k_G) \wedge EG_+ \arrow[r] & F(EG_+, k_G) \arrow[r] & F(EG_+, k_G) \wedge \widetilde{EG}
\end{tikzcd}$$

It commutes by naturality of $F(X, Y) \wedge Z \to F(X, Y \wedge Z)$ in X and Z. As the left and center vertical maps are equivalences, so is the one on the right.

2.2 Filtrations

Filter EG by E_kG, EG_+ by E_kG_+ for $k \geq 0$. In particular, $E_0G = G$, $E_{-1}G = \emptyset$, so $E_0G_+ = G_+, E_{-1}G_+ =$ point. The quotients $R_k = E_kG/E_{k-1}G$ have the form $(\bigvee_{(G-e)^{\times k}} S^k) \wedge G_+$.

Filter $\widetilde{EG} = S(EG)$ (unreduced suspension) by $F_k = \widetilde{E_{k-1}G} = S(E_{k-1}G)$ for $k \geq 0$, so $F_0 = S(\emptyset) = S^0$. Then

$$F_k/F_{k-1} \simeq \Sigma R_{k-1} \simeq \left(\bigvee_{(G-e)^{\times k}} S^k \right) \wedge G_+$$

Extend to $F_{-k} = DF_k$, the Spanier-Whitehead dual of F_k, for $k < 0$. Then

$$F_{-k}/F_{-k-1} = DF_k/DF_{k+1} = \Sigma DR_{k+1} = \left(\bigvee_{(G-e)^{\times (k+1)}} S^{-k} \right) \wedge G_+.$$

The case $G = \mathbb{Z}/2$. Then $RO(G) = \mathbb{Z} \oplus \mathbb{Z}\alpha$ where α denotes the sign representation of $\mathbb{Z}/2$. Take $EG = S(\mathbb{R}^{\infty\alpha})$, $E_kG = S(\mathbb{R}^{(k+1)\alpha})$, a k-sphere. $E_{k-1}G \subset E_kG$ is $S(\mathbb{R}^{k\alpha}) \subset S(\mathbb{R}^{(k+1)\alpha})$, induced by $\mathbb{R}^{k\alpha} \subset \mathbb{R}^{(k+1)\alpha}$.

So $E_kG_+/E_{k-1}G_+ \simeq S^{k\alpha} \wedge \mathbb{Z}/2_+ \simeq S^k \wedge \mathbb{Z}/2_+$. For $k \geq 0$, $F_k = S(E_{k-1}G) = SS(\mathbb{R}^{k\alpha}) \simeq S^{k\alpha}$ (generally $SS(V) = S^V$). $F_{k-1} \subset F_k$ is induced by $\mathbb{R}^{(k-1)\alpha} \subset \mathbb{R}^{k\alpha}$, i.e

$$S^{(k-1)\alpha} = S^{(k-1)\alpha} \wedge S^0 \xrightarrow{1\wedge a} S^{(k-1)\alpha} \wedge S^\alpha = S^{k\alpha}$$

where $a : S^0 \subset S^\alpha$ is induced by $\mathbb{R}^0 \subset \mathbb{R}^\alpha$.

2.3 The Borel and Tate spectral sequences

The Borel spectral sequence for $F(EG_+, k_G)_\star = [EG_+, k_G]_\star$ is constructed from the unrolled exact couple, for $p \geq 0$, with

$$\begin{tikzcd}
D_1^p(c) = [EG_+/E_{p-1}G_+, k_G]_\star \arrow[d] \\
E_1^p(c) = [E_pG_+/E_{p-1}G_+, k_G]_\star \simeq [(\bigvee S^p) \wedge G_+, k_G]_\star = \oplus[G_+, k_G]_{\star+p}
\end{tikzcd}$$

Here, the $\star$ denotes the $RO(G)$-grading.

Use $E_pG_+ \simeq \Sigma^{-1}F_{p+1}$ and Proposition 2.2 to rewrite this as

$$\begin{tikzcd}
D_1^p = [(EG_+/E_{p-1}G_+), k_G]_\star = [\Sigma^{-1}(\widetilde{EG}/F_p), k_G \wedge EG_+]_\star \arrow[d] \\
E_1^p = [\Sigma^{-1}(F_{p+1}/F_p), k_G]_\star \simeq \oplus[G_+, k_G]_{\star+p} = [\Sigma^{-1}(F_{p+1}/F_p), k_G \wedge EG_+]_\star
\end{tikzcd}$$

For $p < 0$ we take $D_1^p = D_1^0, E_1^p = 0$.

This is a half plane spectral sequence, conditionally convergent to $[EG_+, k_G]_\star = k_G^\star(EG_+)$. It is also multiplicative, as EG_+ is filtered by skeletons.

Theorem 2.1. *$E_2^{p,q}(c) = H^p(G; k_G^q)$, for $p \geq 0$ and $q \in RO(G)$.*

Proof $E_1^{p,q}(c) = [R_p, k_G]^{p+q} = \mathrm{Hom}_{\mathbb{Z}[G]}(H_p(R_p), k_G^q)$ for $p \geq 0$, since R_p is a wedge of p-spheres. The differential d_1 is induced by homomorphisms $H_{p+1}(R_{p+1}) \to H_p(R_p)$, which form a resolution of $\mathbb{Z}$ by free $\mathbb{Z}[G]$-modules.

For the Tate spectral sequence we filter $\Sigma^{-1}\widetilde{EG}$ by $\Sigma^{-1}F_{p+1}$, ($p \in \mathbb{Z}$), for

$$t(k_G)_\star = (F(EG_+, k_G) \wedge \widetilde{EG})_\star = [\Sigma^{-1}\widetilde{EG}, k_G \wedge EG_+]_\star.$$

It is defined by the unrolled exact couple (all $p \in \mathbb{Z}$) with

$$\begin{tikzcd}
D_1^p(t) = [\Sigma^{-1}(\widetilde{EG}/F_p), k_G \wedge EG_+]_\star = [\widetilde{EG}/F_p, k_G \wedge EG_+]_{\star-1} \arrow[d] \\
E_1^p(t) = [\Sigma^{-1}(F_{p+1}/F_p), k_G \wedge EG_+]_\star = [\Sigma^{-1}(F_{p+1}/F_p), k_G]_\star = \oplus[G_+, k_G]_{\star+p}
\end{tikzcd}$$

This gives a whole plane spectral sequence, conditionally convergent to $\mathrm{colim}_p D_1^p$; it is a multiplicative spectral sequence, since $\Sigma^{-1}\widetilde{EG}$ is filtered by the skeletons. We extend the above resolution of $\mathbb{Z}$ by defining $R_p = \Sigma^{-1}(F_{p+1}/F_p)$ for all $p \in \mathbb{Z}$. Given any $\mathbb{Z}[G]$-module M, the cohomology groups of the chain complex of groups $\mathrm{Hom}_{\mathbb{Z}[G]}(H_p(R_p), M)$ are known as the *Tate cohomology* groups $\widehat{H}^p(G; M)$ of G with coefficients in M.

Theorem 2.2. $E_2^{p,q}(t) = \widehat{H}^p(G; k_G^q)$*, for* $p \in \mathbb{Z}$ *and* $q \in RO(G)$*.*

2.4 A morphism of spectral sequences

The filtered map $\Sigma^{-1}\widetilde{EG} \to \Sigma^{-1}(\widetilde{EG}/S^0)$ induces a morphism of spectral sequences from the Borel spectral sequence to the Tate spectral sequence. In more detail, $E_1^p(c) = E_1^p(t)$ for $p \geq 0$, $E_1^p(c) = 0$ for $p < 0$ and

$$\begin{tikzcd}
\cdots \arrow[r] & D_1^2(c) \arrow[r] \arrow[d, "="'] & D_1^1(c) \arrow[r] \arrow[d, "="'] & D_1^0(c) \arrow[r, "="] \arrow[d, "="'] & D_1^{-1}(c) \arrow[r, "="] \arrow[d] & D_1^{-2}(c) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & D_1^2(t) \arrow[r] & D_1^1(t) \arrow[r] & D_1^0(t) \arrow[r] & D_1^{-1}(t) \arrow[r] & D_1^{-2}(t) \arrow[r] & \cdots
\end{tikzcd}$$

The ring structures of $F(\Sigma^{-1}\widetilde{EG}, k_G \wedge EG_+)$ and $F(\Sigma^{-1}(\widetilde{EG}/S^0), k_G \wedge EG_+)$ are inherited from $c(k_G)$ and $t(k_G)$ from Proposition 2.1.

Our third spectral sequence is again obtained by filtering EG_+. We use the unrolled exact couple with

$$\begin{tikzcd}
D^1_p(f) \arrow[r, equal] & {[S, k_G \wedge E_pG_+]_\star} \arrow[d] \\
E^1_p(f) \arrow[r, equal] & {[S, k_G \wedge R_p]_\star}
\end{tikzcd}$$

for $p \geq 0$, also $D^1_p(f) = D^1_0(f)$ and $E^1_p(f) = 0$ for $p < 0$.

Since R_p is a wedge of p-spheres, we may write

$$E^1_{p,q}(f) = [S, k_G \wedge R_p]_{p+q} = H_p(R_p) \otimes (k_G)_q$$

for $p \geq 0$.

Theorem 2.3. *$E^2_{p,q}(f) = H_p(G; k^G_q)$ for $p \geq 0$ and $q \in RO(G)$.*

There is a map, using Spanier-Whitehead duality for $p < 0$, from the Tate spectral sequence to this spectral sequence, induced by the following morphism of unrolled exact couples.

$$\begin{tikzcd}
D_1^{p,q}(t) \arrow[r] \arrow[d, equal] & E_1^{p,q}(t) \arrow[r] \arrow[d, equal] & D_1^{p+1,q}(t) \arrow[d, equal] \\
{[\Sigma^{-1}\frac{\widetilde{EG}}{F_p}, k_G \wedge EG_+]^{p+q}} \arrow[r] \arrow[d] & {[\Sigma^{-1}\frac{F_{p+1}}{F_p}, k_G \wedge EG_+]^{p+q}} \arrow[r, "\delta^*"] \arrow[d, "\simeq"'] & {[\Sigma^{-1}\frac{\widetilde{EG}}{F_{p+1}}, k_G \wedge EG_+]^{p+q+1}} \arrow[d] \\
{[\Sigma^{-1}\frac{S^0}{F_p}, k_G]^{p+q}} \arrow[r] \arrow[d, equal] & {[\Sigma^{-1}\frac{F_{p+1}}{F_p}, k_G]^{p+q}} \arrow[r, "\delta^*"] \arrow[d, equal] & {[\Sigma^{-1}\frac{S^0}{F_{p+1}}, k_G]^{p+q+1}} \arrow[d, equal] \\
k_G^{p+q}\left(\Sigma^{-1}\frac{S^0}{F_p}\right) \arrow[r] \arrow[d, "="'] & k_G^{p+q}\left(\Sigma^{-1}\frac{F_{p+1}}{F_p}\right) \arrow[r, "\delta^*"] \arrow[d, "="'] & k_G^{p+q+1}\left(\Sigma^{-1}\frac{S^0}{F_{p+1}}\right) \arrow[d, "="'] \\
k^G_{-p-q}\left(\frac{F_{-p}}{S^0}\right) \arrow[r] \arrow[d, "="'] & k^G_{-p-q}\left(\frac{F_{-p}}{F_{-p-1}}\right) \arrow[r, "\delta_*"] \arrow[d, "="'] & k^G_{-p-q-1}\left(\frac{F_{-p-1}}{S^0}\right) \arrow[d, "="'] \\
k^G_{-p-q}(\Sigma E_{-p-1}G_+) \arrow[r] \arrow[d, equal] & k^G_{-p-q}\left(\Sigma\frac{E_{-p-1}G_+}{E_{-p-2}G_+}\right) \arrow[r, "\delta_*"] \arrow[d, equal] & k^G_{-p-q-1}(\Sigma E_{-p-2}G_+) \arrow[d, equal] \\
k^G_{-p-q-1}(E_{-p-1}G_+) \arrow[r] \arrow[d, equal] & k^G_{-p-q-1}\left(\frac{E_{-p-1}G_+}{E_{-p-2}G_+}\right) \arrow[r, "\delta_*"] \arrow[d, equal] & k^G_{-p-q-2}(E_{-p-2}G_+) \arrow[d, equal] \\
D^1_{-p-1,-q}(f) \arrow[r] & E^1_{-p-1,-q}(f) \arrow[r] & D^1_{-p-2,-q}(f)
\end{tikzcd}$$

For $p \geq 0$, use $D_1^p(t) \to D_1^{-1}(t) \to D_0^1(f) = D^1_{-p-1}(f)$, $E^1_{-p-1}(f) = 0$

There is a short exact sequence

$$0 \to E_1^p(c) \to E_1^p(t) \to E^1_{-p-1}(f) \to 0$$

for all p, where $E_1^p(c) = 0$ for $p < 0$ and $E^1_{-p-1}(f) = 0$ for $p \geq 0$.

We summarize the relation between Tate cohomology and group (co)homology.

Corollary 2.1. *For any $\mathbb{Z}[G]$-module M,*

$$\widehat{H}^p(G; M) = \begin{cases} H^p(G; M) & \textit{for } p \geq 1 \\ H_{-p-1}(G; M) & \textit{for } p \leq -2 \end{cases}$$

3 Real-oriented spectra

We shall review the theory of Real cobordism and Real-oriented spectra discovered by Landweber and Araki ([1],[15]). All our spectra will be $\mathbb{Z}/2$-equivariant.

Let $\mathbb{S}^1$ denote the unit circle group $S^1 \subset \mathbb{C}^*$ with complex conjugation as involution and a basepoint 1. A *Real space* in the sense of Atiyah [2] is space with an involution. Let $B\mathbb{S}^1$ be the classifying space of S^1, which is the space of all complex lines in $\mathbb{C}^\infty$ with complex conjugation as involution. We have $\Omega B\mathbb{S}^1 \simeq \mathbb{S}^1$ in the category of based $\mathbb{Z}/2$-spaces. Thus, by adjunction we have a canonical equivariant based map

$$\eta : S^{1+\alpha} \to B\mathbb{S}^1$$

where α is the non-trivial one-dimensional representation of $\mathbb{Z}/2$.

Definition 3.1. *Let E be a $\mathbb{Z}/2$-equivariant commutative associative ring spectrum. A Real orientation of E is a map*

$$u : B\mathbb{S}^1 \to \Sigma^{1+\alpha} E$$

which makes the following diagram commute

$$\begin{tikzcd}
S^{1+\alpha} \arrow[r, "\eta"] \arrow[dr, "1"'] & B\mathbb{S}^1 \arrow[d, "u"] \\
& \Sigma^{1+\alpha} E
\end{tikzcd}$$

First we describe the $\mathbb{Z}/2$-spectrum $M\mathbb{R}$ that represents Real complex cobordism. Let $MU(n)$ denote the Thom space of the universal bundle γ_n over $BU(n)$. Complex conjugation induces an action of $\mathbb{Z}/2$ on $MU(n)$. The space $M\mathbb{U}(n)$ (with conjugation action) is placed in dimension $n(1+\alpha)$. The canonical Real bundle γ_n of dimension

n over $BU(n)$ gives maps between Thom spaces,

$$\Sigma^{1+\alpha} BU(n)^{\gamma_n} \to BU(n+1)^{\gamma_{n+1}}$$

which gives our required structure maps. Spectrification makes $M\mathbb{R}$ a $\mathbb{Z}/2$-spectrum. Recall that $RO(\mathbb{Z}/2) = \mathbb{Z} \oplus \mathbb{Z}\alpha$ where α represents the sign representation. For V in $RO(\mathbb{Z}/2)$ define $M\mathbb{R}(V)$ as $\operatorname{colim} \Omega^{n(1+\alpha)-V} M\mathbb{U}(n)$. $M\mathbb{R}$ is a multiplicative Real-oriented spectrum in the sense of [9]. We shall write the coefficient ring $M\mathbb{R}_\star$, the $\star$ denoting the $RO(\mathbb{Z}/2)$-grading.

Proposition 3.1. *The spectrum $M\mathbb{R}$ is Real-oriented.*

Proof As in the complex case, $B\mathbb{S}^1$ is the Thom space of the canonical Real bundle on $B\mathbb{S}^1$, and therefore the $(1+\alpha)$-term of the prespectrum defining $M\mathbb{R}$.

Proposition 3.2. *The spectrum KR representing Atiyah's Real K-theory is Real-oriented.*

Proof The tensor product of Real bundles makes KR into a commutative associative ring spectrum. The 0th space of the spectrum KR is $\mathbb{Z} \times B\mathbb{U}$. The Real Bott periodicity gives an equivalence of spectra

$$\Sigma^{1+\alpha} KR \to KR.$$

By a Real CW-complex we will mean a $\mathbb{Z}/2$-equivariant space $K = \bigcup_{n \geq 0} K_n$ where K_0 is a discrete space and K_n is obtained by the pushout diagram

$$\begin{tikzcd}
\coprod \mathbb{S}^{2n-1} \arrow[r] \arrow[d] & K_{n-1} \arrow[d] \\
\coprod \mathbb{D}^{2n} \arrow[r] & K_n
\end{tikzcd}$$

where $\mathbb{S}^{2n-1} = S(\mathbb{C}^n)$ and $\mathbb{D}^n = D(\mathbb{C}^n)$. The filtration $K_0 \subset K_1 \subset K_2 \subset \ldots \subset K$ is the Real filtration of K. A map of Real spectra is a $\mathbb{Z}/2$-equivariant map that preserves the Real filtration.

Now let E be a $\mathbb{Z}/2$-equivariant spectrum and let K be a Real CW-complex. Then analogous to the non-equivariant case there are cofibrations,

$$K_{n-1} \to K_n \to \bigvee_{\text{Real}\, n-\text{cells}} S^{n(1+\alpha)}.$$

Applying E-cohomology we get an exact couple which gives a conditionally convergent spectral sequence

$$E_1^{p,q} = \oplus_{\text{Real}\, p-\text{cell of}\, X} E^q \Rightarrow E^{p(1+\alpha)+q}(K)$$

where $q \in RO(\mathbb{Z}/2)$. If E is a ring spectrum then the spectral sequence has a ring structure. We see that $B\mathbb{S}^1$ is a Real CW-spectrum. $E_1 = E^\star[[u]]$ for the spectral sequence given above for $K = B\mathbb{S}^1$, and if E is Real-oriented u is a permanent cycle. Similarly for $K = B\mathbb{S}^1 \times B\mathbb{S}^1$.

Theorem 3.1. *If E is a Real-oriented spectrum, then*

$$E^\star(B\mathbb{S}^1) = E^\star[[u]],\, \dim(u) = -(1+\alpha)$$

$$E^\star(B\mathbb{S}^1 \times B\mathbb{S}^1) = E^\star[[u \otimes 1, 1 \otimes u]].$$

Therefore the map on classifying spaces induced by the product $\mathbb{S}^1 \times \mathbb{S}^1 \to \mathbb{S}^1$ gives a formal group law over the ring $E_\star$.

The grading in $E_\star$ is such that the formal group law $F(x, y)$ is homogeneous of degree $-(1+\alpha)$ if both x and y are of degree $-(1+\alpha)$. Also isomorphisms of formal group laws are formal power series over x of degree $-(1+\alpha)$ where x is of dimension $-(1+\alpha)$.

If E is a Real-oriented spectrum, then multiplication on $E_\star$ is graded commutative. If $x \in E_{k+l\alpha}$ and $y \in E_{m+n\alpha}$, then

$$xy = (-1)^{km+ln} yx \in E_{(k+m)+(l+n)\alpha}.$$

Thus, $E_{*(1+\alpha)} \subset E_\star$ is a commutative ring.

By Theorem 3.1 and the fact that MU_* is the universal formal group law, we get

Proposition 3.3. *The canonical map of rings $M\mathbb{R}_\star \to MU_*$ splits by a map of rings $MU_* \to M\mathbb{R}_\star$ which sends the generator $x_i \in MU_{2i}$ to an element in degree $i(1+\alpha)$.*

Theorem 3.2. *For every Real-oriented spectrum E, every Real bundle is E-oriented. Moreover, $E^\star(M\mathbb{R}) = E^\star[b_1, b_2, \ldots]$, and there is a bijective correspondence between Real orientations of E and maps of ring spectra $M\mathbb{R} \to E$, which again are in bijective correspondence with strict isomorphisms of formal group laws whose source is the formal group law over $E_\star$.*

Following this theorem we obtain a Real-oriented spectrum $BP\mathbb{R}$ using Quillen idempotents. The localized spectrum $M\mathbb{R}_{(p)}$ splits as

$$M\mathbb{R}_{(p)} = \bigvee \Sigma^{m_i(1+\alpha)} BP\mathbb{R}$$

3.1 $E\mathbb{R}(n)$

Here we will use the E_∞-module theory of [5] and construct Real analogues of complex-oriented spectra. We shall construct $E\mathbb{R}(n)$ as a Real-oriented spectrum analogue of the complex oriented $E(n)$. First we need the fact:

Proposition 3.4. *$M\mathbb{R}$ is a an E_∞-ring spectrum.*

Now we have, directly by construction, the underlying non-equivariant spectrum of $M\mathbb{R}$

$$F(\mathbb{Z}/2_+, M\mathbb{R})^{\mathbb{Z}/2} = MU.$$

Let $MU_* = \mathbb{Z}[x_1, \ldots, x_n, \ldots]$. By a derived spectrum of MU we mean an E_∞-MU-module obtained by killing off a regular sequence

$$(z_1, z_2, \ldots) \in MU_*$$

and localizing at elements in MU_*.

By Proposition 3.3 we have a map $\mathbb{Z}[x_1, \ldots, x_n, \ldots] \to M\mathbb{R}_\star$. Therefore for a derived spectrum E of MU we can construct an E_∞-$M\mathbb{R}$-module $\mathbb{E}$ by killing off and localizing at the images in $M\mathbb{R}_\star$ of the relevant elements of MU_*. In more detail, for every element $z \in M\mathbb{R}_\star$ and every E_∞-$M\mathbb{R}$-module $\mathbb{M}$ there is a self-map of $M\mathbb{R}$ modules $z : \mathbb{M} \to \mathbb{M}$. If we denote by the z_i the images in $M\mathbb{R}_\star$ of the elements $z_i \in MU_*$ and let $M_0 = M\mathbb{R}$, we obtain the E_∞-$M\mathbb{R}$-module $\mathbb{M}_i = \mathbb{M}/(z_1, \ldots, z_i)$ using the standard cofibration sequences of $M\mathbb{R}$ modules:

$$\begin{tikzcd}\mathbb{M}_{i-1} \arrow[r, "z_i"] & \mathbb{M}_{i-1} \arrow[r] & \mathbb{M}_i\end{tikzcd}$$

The localization $\mathbb{M}[z^{-1}]$ is defined as the colimit in the category of E_∞-$M\mathbb{R}$-modules of the diagram

$$\begin{tikzcd}\mathbb{M} \arrow[r, "z"] & \mathbb{M} \arrow[r, "z"] & \mathbb{M} \arrow[r, "z"] & \cdots\end{tikzcd}$$

Observe that the lifts of z_i do not necessarily form a regular sequence in $M\mathbb{R}_\star$. This makes the calculation of the $\mathbb{M}_\star$ non-trivial. However the underlying non-equivariant spectrum of $\mathbb{E}$ is given by $F(\mathbb{Z}/2_+, \mathbb{E})^{\mathbb{Z}/2} = E$. This will enable us in some cases, where there is a strong completion, to compute $\mathbb{E}_\star$ using the Borel spectral sequence.

3.2 The elements $\sigma^{\pm 1}$ and a

There is an obvious homeomorphism of spaces $S^\alpha \wedge \mathbb{Z}/2_+ \simeq S^1 \wedge \mathbb{Z}/2_+$. In the Borel spectral sequence, $E_1^0 = [(E_o\mathbb{Z}/2)_+, E\mathbb{R}(n)]_\star = [\mathbb{Z}/2_+, E\mathbb{R}(n)]_\star$, define $\sigma \in E_1^0$ as

$$\Sigma^{\alpha-1}\mathbb{Z}/2_+ = \Sigma^{-1}(S^\alpha \wedge \mathbb{Z}/2_+) \simeq \Sigma^{-1}(S^1 \wedge \mathbb{Z}/2_+) = \mathbb{Z}/2_+ \to S^0 \to E\mathbb{R}(n)$$

so $\sigma^{-1} \in E_1^0$ is

$$\Sigma^{1-\alpha}\mathbb{Z}/2_+ = \Sigma^{-\alpha}(S^1 \wedge \mathbb{Z}/2_+) \simeq \Sigma^{-\alpha}(S^\alpha \wedge \mathbb{Z}/2_+) = \mathbb{Z}/2_+ \to S^0 \to E\mathbb{R}(n),$$

using the unit map $S^0 \to E\mathbb{R}(n)$.

We write a for the class of the inclusion $a : S^0 \subset S^\alpha$ induced by $\mathbb{R}^0 \subset \mathbb{R}^\alpha$. It acts on everything, i.e on $[X, Y]_\star$, all X, Y, naturally in X and Y. In particular, it acts on the element $1 \in [E\mathbb{Z}/2_+, E\mathbb{R}(n)]_\star$ to produce the element $a = a.1 \in D_1^0$ of degree $-\alpha$, which may be written

$$a.1 : S(\mathbb{R}^{\infty\alpha})_+ \longrightarrow S^0 \xrightarrow{a} S^\alpha \xrightarrow{\Sigma^\alpha \eta} \Sigma^\alpha E\mathbb{R}(n)$$

Since $\mathbb{Z}/2_+ \to S^0 \to S^\alpha$ is obviously zero, $a.1$ maps to $0 \in E_1^0$ and therefore lifts to D_1^1. Further, this lift is canonical: the orthogonal projection $P : \mathbb{R}^{\infty\alpha} \to \mathbb{R}^\alpha$ induces a map $P : (S(\mathbb{R}^{\infty\alpha}), S(\mathbb{R}^\alpha)) \to (D(\mathbb{R}^\alpha), S(\mathbb{R}^\alpha))$ and we may describe the lift as the element

$$S(\mathbb{R}^{\infty\alpha})_+/S(\mathbb{R}^\alpha)_+ \xrightarrow{P_+} D(\mathbb{R}^\alpha)_+/S(\mathbb{R}^\alpha)_+ \xrightarrow{\simeq} S^\alpha \xrightarrow{=} \Sigma^\alpha S^0 \xrightarrow{\Sigma^\alpha \eta} \Sigma^\alpha E\mathbb{R}(n)$$

of $D_1^1(c)$. This maps to an element of $E_1^1(c)$, of degree $-\alpha$, that we also call a.

3.3 The Tate spectral sequence for $E\mathbb{R}(n)$

For the E_1-term we have

$$E_1^{p,q}(t) = [R_p, E\mathbb{R}(n)]^{p+q} = [\mathbb{Z}/2_+, E\mathbb{R}(n)]^q = E(n)^q$$

where $q \in RO(\mathbb{Z}/2)$. We have the element $a \in E_1^1$ and the invertible element $\sigma \in E_1^0$ of degrees $-\alpha$ and $\alpha - 1$, introduced above. Further, one can show that multiplication by a induces an isomorphism $E_1^{p,q} \simeq E_1^{p+1,q-1+\alpha}$ for all $p \in \mathbb{Z}$ and $q \in RO(\mathbb{Z}/2)$. This is all there is:

$$E_1(t) = E(n)_*[a^{\pm 1}, \sigma^{\pm 1}]$$

where $E(n)_* = \mathbb{Z}_{(2)}[v_1, v_2, \ldots, v_n^{\pm 1}]$ and $\deg v_i = (2^i - 1)(1 + \alpha)$. We also write $v_0 = 2$. We have a spectral sequence of $E(n)_*$-modules.

Following [8], the differential d_1 is determined by $d_1(\sigma^{-1}) = v_0 a$, with the result that

$$E_2(t) = \mathbb{F}_2[v_1, v_2, \ldots, v_n^{\pm 1}][\sigma^{\pm 2}, a^{\pm 1}]$$

(the exponent of a simply indicates the filtration). From the E_2-term on we have a spectral sequence of graded commutative rings, and each d_r is a derivation.

We proceed by induction on s, for $1 \le s < n$, and assume

$$E_{2^s}(t) = \mathbb{F}_2[v_s, v_{s+1}, \ldots, v_n^{\pm 1}][\sigma^{\pm 2^s}, a^{\pm 1}]. \tag{4}$$

By sparseness, $d_r = 0$ for $2^s \le r \le 2^{s+1} - 2$. By [9], the next differential is determined by $d_{2^{s+1}-1}(\sigma^{-2^s}) = v_s a^{2^{s+1}-1}$, so that the non-zero differentials are

$$d_{2^{s+1}-1}(v^R \sigma^{2^s(2l-1)} a^k) = v_s v^R \sigma^{2^{s+1}l} a^{k+2^{s+1}-1} \qquad (l \in \mathbb{Z}, k \in \mathbb{Z}, 0 \le s \le n)$$

where $v^R = v_s^{r_s} v_{s+1}^{r_{s+1}} \ldots v_n^{r_n}$. The ideal (v_s) is killed and we find $E_{2^{s+1}}(t)$ as (4) above, with s replaced by $s+1$.

Then $E_{2^n}(t) = \mathbb{F}_2[v_n^{\pm 1}][\sigma^{\pm 2^n}, a^{\pm 1}]$. By the same argument, $d_{2^{n+1}-1}$ kills (v_n); but v_n is invertible, so $E_{2^{n+1}}(t) = 0$. Thus the spectral sequence converges strongly to 0. We would like to conclude that $t(E\mathbb{R}(n))$ is trivial. Unfortunately, this is a whole plane spectral sequence with mysterious target, and we have to proceed indirectly.

3.4 The Borel spectral sequence for $E\mathbb{R}(n)$

We have a morphism of spectral sequences $\omega_r : E_r(c) \to E_r(t)$ of $E(n)_*$-modules. The first lemma is easily proved.

Lemma 3.1. *$\omega_r : E_r^p(c) \to E_r^p(t)$ is*

1. *surjective for $p \geq 0$*
2. *bijective for $p \geq r-1$*

Corollary 3.1. *For $x \in E_r^p(c)$, $\omega_r x = 0 \Rightarrow d_r x = 0$.*

Proof $d_r x \in E_r^{p+r}$. $\omega_r d_r x = d_r \omega_r x = 0$, therefore $d_r x = 0$.

Corollary 3.2. *$d_r = 0$ in $E_r(c)$ unless $r+1$ is a power of 2.*

Proof Since true in $E_r(t)$.

Therefore we have a filtration

$$0 \subset \mathrm{Ker}\omega_2 \subset \mathrm{Ker}\omega_4 \subset \ldots \subset \mathrm{Ker}\omega_{2^{n+1}} = E_{2^{n+1}}(c) = E_\infty(c)$$

by $E(n)_*$-modules. $\mathrm{Ker}\omega_2$ is a free $E(n)_*$-module on generators $v_0\sigma^{2s} = 2\sigma^{2s} (s \in \mathbb{Z})$. Generally for $0 \leq s \leq n$, $\mathrm{Ker}\omega_{2^s} \subset \mathrm{Ker}\omega_{2^{s+1}}$ introduces new $E(n)_*$-module generators $v_s\sigma^{2^{s+1}l}a^u$ $(l \in \mathbb{Z}, 0 \leq u \leq 2^{s+1}-2)$. This gives new abelian group generators

$v_s v^R \sigma^{2^{s+1}l} a^u$ from $d_{2^{s+1}-1}(\sigma^{2^s(2l-1)} a^{u-2^{s+1}+1}) = v_s \sigma^{2^{s+1}l} a^u$ in $E_{2^{s+1}-1}(t)$ which yield a basis over $\mathbb{F}_2$ of $\mathrm{Ker}\omega_{2^{s+1}}/\mathrm{Ker}\omega_{2^s}$.

We first describe $E_\infty(c)$ as a filtered module over $E(n)_*[a]$. If $b \in E_1(c)$ is a permanent cycle, we write $[b]$ for its class in $E_\infty(c)$. As a module, it is generated by the elements

$$[v_s \sigma^{2^{s+1}} l] \ \ (0 \leq s \leq n, l \in \mathbb{Z}) \tag{5}$$

with module relations

$$a^{2^{s+1}-1}[v_s \sigma^{2^{s+1}l}] = 0, \qquad v_i[v_s \sigma^{2^{s+1}l}] = v_s[v_i \sigma^{2^{i+1}(2^{s-i}l)}] \ \ (\text{for } i < s) \tag{6}$$

It is often useful to replace the generator $[v_n \sigma^{2^{n+1}(0)}]$ by $v_n^{-1}[v_n \sigma^{2^{n+1}(0)}]$, the identity element of the ring $E_\infty(c)$. More generally we can replace $[v_n \sigma^{2^{n+1}l}]$ by $v_n^{-1}[v_n \sigma^{2^{n+1}l}] = [\sigma^{2^{n+1}l}]$. The degrees of a, v_k and σ are $-\alpha$,$(2^k-1)(1+\alpha)$ and $(\alpha-1)$ respectively.

Multiplication (which is commutative) is given on the generators by

$$[v_s \sigma^{2^{s+1}l}][v_m \sigma^{2^{m+1}q}] = v_m[v_s \sigma^{2^{s+1}(l+2^{m-s}q)}] \ (\text{for } s \leq m) \tag{7}$$

This is clearly inherited from $E_1(c)$. In particular, since $v_0 = 2$, $[v_0\sigma^{2l}][v_0\sigma^{2q}] = 2[v_0\sigma^{2(l+q)}]$.

We really want $c(E\mathbb{R}(n))_\star$, by unfiltering $E_\infty(c)$. By sparseness of the spectral sequence, when $n = 2$, each generator $[v_s \sigma^{2^{s+1}} l]$ lifts *uniquely* to an element of $c(E\mathbb{R}(2))_\star$ that we also call $[v_s \sigma^{2^{s+1}} l]$; further, the module relations (6) also lift uniquely to $c(E\mathbb{R}(2))_\star$. However even for $n = 2$, a few of the multiplications (7) do not lift uniquely; there is some indeterminacy, but we do have $[\sigma^{2^{n+1}l}][\sigma^{2^{n+1}q}] = [\sigma^{2^{n+1}}(l+q)]$ in $c(E\mathbb{R}(n))_\star$ for any n.

We follow [11] and define the element $y(n) = v_n^{2^n-1}\sigma^{-2^{n+1}(2^{n-1}-1)}$ in degree $\lambda(n)+\alpha$ where $\lambda(n) = 2^{2n+1} - 2^{n+2} + 1$. The element $y(n)$ is clearly invertible in the $RO(\mathbb{Z}/2)$-

graded ring $c(E\mathbb{R}(n))_\star$. We shall use it to reduce elements of $c(E\mathbb{R}(n))_\star$ to elements with integer degree. Define $x(n) = y(n)a$, which has degree $\lambda(n)$.

3.5 Strong completion and cofibrations

Assume given a representation V of G whose unit sphere $S(V)$ is a free G-space. The union $S(\infty V)$ of the spheres $S(qV)$ is a model for EG, the union $D(\infty V)$ of unit discs $D(qV)$ is G-contractible, and the quotient $D(\infty V)/S(\infty V) \simeq S^{\infty V}$ is a model for $\widetilde{EG}$.

In homology and cohomology, the description of $\widetilde{EG}$ as $S^{\infty V}$ implies the following canonical isomorphisms for a G-spectrum k_G, an integer n (or more general element of $RO(G)$, and a G-space X, where X is finite in the case of cohomology:

$$(k_G \wedge \widetilde{EG})_n(X) = \mathrm{colim}_q (k_G \wedge S^{qV})_n(X) = \mathrm{colim}_q k^G_{n-qV}(X)$$

and

$$(k_G \wedge \widetilde{EG})^n(X) = \mathrm{colim}_q (k_G \wedge S^{qV})^n(X) = \mathrm{colim}_q k_G^{n+qV}(X)$$

The colimits can be interpreted algebraically. Let V be any representation of G, let $e : S^0 \to S^V$ be induced by $0 \subset V$, and let $a_V \in k_G^V(S^0)$ be the image of the identity element of $k_G^0(S^0)$ under the map $e^* : k_G^0(S^0) = k_G^V(S^V) \to k_G^V(S^0)$. Therefore given a ring spectrum k_G and any G-representation V, $(k_G \wedge S^{\infty V})_\star(X)$ and, if X is finite, $(k_G \wedge S^{\infty V})^\star(X)$, are the localizations of $k^G_\star(X)$ and $k_G^\star(X)$ away from a_V. Thus multiplication by a_V provides a periodicity isomorphism with period V on $(k_G \wedge S^{\infty V})_\star(X)$ and $(k_G \wedge S^{\infty V})^\star(X)$.

Following Proposition 2.1 this has the following implication.

Proposition 3.5. *Let k_G be a ring G-spectrum. If G acts freely on the unit sphere $S(V)$, then $t(k_G)_\star(X)$ and, if X is finite, $t(k_G)^\star(X)$ are localizations of $c(k_G)_\star(X)$*

and $c(k_G)^\star(X)$ *away from* a_V*. Therefore, multiplication by* a_V *provides a periodicity isomorphism with period* V *on* $t(k_G)_\star(X)$ *and* $t(k_G)^\star(X)$*.*

In the case of $E\mathbb{R}(n)$, $G = \mathbb{Z}/2$, $V = \alpha$, $a_V = a$, and there is a commutative square of $\mathbb{Z}/2$-equivariant ring spectra:

$$\begin{tikzcd}
E\mathbb{R}(n) \arrow[r] \arrow[d] & g(E\mathbb{R}(n)) = \widetilde{E\mathbb{Z}/2} \wedge E\mathbb{R}(n) \arrow[d] \\
c(E\mathbb{R}(n)) \arrow[r] & t(E\mathbb{R}(n))
\end{tikzcd}$$

The element $a : S^0 \subset S^\alpha$ acts

1. nilpotently on $c(E\mathbb{R}(n))_\star$.
2. invertibly on $g(E\mathbb{R}(n))_\star$ and $t(E\mathbb{R}(n))_\star$.

Theorem 3.3. *The Tate spectrum* $t(E\mathbb{R}(n))$ *is trivial.*

Proof It is the localization of $c(E\mathbb{R}(n))$ away from a, so on $t(E\mathbb{R}(n))_\star$, a acts invertibly as well as nilpotently.

Since the left vertical arrow of the Tate diagram is an equivalence, we see that, up to equivalence, $t(k_G)$ can be seen as the cofiber of the composite

$$k_G \wedge EG_+ \to k_G \to F(EG_+, k_G).$$

This gives us the following result.

Proposition 3.6. $f(E\mathbb{R}(n)) \simeq c(E\mathbb{R}(n))$.

We will construct a cofibration of spectra connecting $E(n)$ and $ER(n)$. We have the following equivalence as part of the Tate diagram.

$$\begin{tikzcd}
f(E\mathbb{R}(n)) \arrow[r] \arrow[dr, "\simeq"] & E\mathbb{R}(n) \arrow[r] \arrow[d] & g(E\mathbb{R}(n)) \\
 & c(E\mathbb{R}(n)) &
\end{tikzcd}$$

This implies a splitting of $\mathbb{Z}/2$- ring spectra

$$E\mathbb{R}(n) \simeq c(E\mathbb{R}(n)) \vee g(E\mathbb{R}(n)).$$

There is a fibration $\mathbb{Z}/2_+ \longrightarrow S^0 \xrightarrow{a} S^\alpha$ inducing a fibration

$$\begin{tikzcd}
F(S^\alpha, g(E\mathbb{R}(n))) \arrow[r, "a^*"] \arrow[d, "\simeq"] & F(S^0, g(E\mathbb{R}(n))) \arrow[r] \arrow[d, "\simeq"] & F(\mathbb{Z}/2_+, g(E\mathbb{R}(n))) \\
\Sigma^{-\alpha} g(E\mathbb{R}(n)) \arrow[r, "a", "\simeq"'] & g(E\mathbb{R}(n)) &
\end{tikzcd}$$

This equivalence induced by a implies that $F(\mathbb{Z}/2_+, g(E\mathbb{R}(n)))$ is trivial. We have the analogous fibration

$$\begin{tikzcd}
\Sigma^{-\alpha} c(E\mathbb{R}(n)) \arrow[r, "a"] & c(E\mathbb{R}(n)) \arrow[r] & F(\mathbb{Z}/2_+, c(E\mathbb{R}(n))) \arrow[d, "="] \\
\Sigma^{\lambda(n)} c(E\mathbb{R}(n)) \arrow[u, "y(n)", "\simeq"'] \arrow[ur, "x(n)"'] & & F(\mathbb{Z}/2_+, E\mathbb{R}(n))
\end{tikzcd}$$

On fixed points we deduce the fibration of non-equivariant spectra

$$\Sigma^{\lambda(n)} ER(n) \xrightarrow{x(n)} ER(n) \longrightarrow E(n) \tag{8}$$

where $ER(n) := E\mathbb{R}(n)^{h\mathbb{Z}/2}$, the homotopy fixed points of $E\mathbb{R}(n)$, which is just the ordinary fixed points of the Borel spectrum $c(E\mathbb{R}(n))$. The element $x(n)$ has degree $\lambda(n) = 2^{2n+1} - 2^{n+2} + 1$.

3.6 The boundary map

We shall now compute the boundary map

$$\partial : \Sigma^{-1}E(n) \to \Sigma^{\lambda(n)}ER(n) \to \Sigma^{\lambda(n)}E(n)$$

Consider the composite $\Sigma^{\alpha}\mathbb{Z}/2_{+} \to S^{\alpha} \to \Sigma\mathbb{Z}/2_{+}$ Notice that multiplication by σ^{-1} induces a non-equivariant equivalence $\Sigma^{-\alpha}E\mathbb{R}(n) \to \Sigma^{-1}E\mathbb{R}(n)$, whereas multiplication by $y(n)$ induces an equivalence of $\mathbb{Z}/2$-spectra $\Sigma^{-\alpha}E\mathbb{R}(n) \simeq \Sigma^{\lambda(n)}E\mathbb{R}(n)$.

This induces the following diagram of naive non-equivariant spectra:

$$\begin{tikzcd}
F(\mathbb{Z}/2_{+}, \Sigma^{-1}E(n)) \arrow[r, "(-)"] & F(S, \Sigma^{-1}E(n)) \arrow[r, "\Delta"] & F(\mathbb{Z}/2_{+}, \Sigma^{-1}E(n)) \\
i^{*}F(\mathbb{Z}/2_{+}, \Sigma^{-1}E\mathbb{R}(n)) \arrow[r] \arrow[u, "\simeq"] & i^{*}F(S, \Sigma^{-\alpha}E\mathbb{R}(n)) \arrow[r] \arrow[u, "\simeq", "\sigma^{-1}"'] & i^{*}F(\mathbb{Z}/2_{+}, \Sigma^{-\alpha}E\mathbb{R}(n)) \arrow[u, "\simeq", "\sigma^{-1}"'] \\
F(\mathbb{Z}/2_{+}, \Sigma^{-1}E\mathbb{R}(n))^{\mathbb{Z}/2} \arrow[r] \arrow[u, hook, "{(1,c)}"] & F(S^{\alpha}, E\mathbb{R}(n))^{\mathbb{Z}/2} \arrow[r] \arrow[u, hook] & F(\mathbb{Z}/2_{+}, \Sigma^{-\alpha}E\mathbb{R}(n))^{\mathbb{Z}/2} \arrow[u, hook, "{(1,c)}"] \\
\Sigma^{-1}E(n) \arrow[r] \arrow[u, "\simeq"] & \Sigma^{\lambda(n)}ER(n) \arrow[r] \arrow[u, "\simeq", "y(n)"'] & \Sigma^{\lambda(n)}E(n) \arrow[u, "\simeq", "y(n)"']
\end{tikzcd}$$

where $(-)$ is the difference map, Δ is the twisted diagonal map and c is complex conjugation. It follows from above that

Proposition 3.7. $\partial = y(n)^{-1}\sigma(1-c)$

4 Cohomology of projective spaces

Consider the complex projective space $\mathbb{C}P^{\infty}$, with the action of $\mathbb{Z}/2$ given by complex conjugation. Consider the canonical line bundle γ on $\mathbb{C}P^{\infty}$. This is a Real bundle in Atiyah's sense [2]. Recall that our theory is Real-oriented [9], which means there is a Real Chern class $u \in E\mathbb{R}(n)^{1+\alpha}(\mathbb{C}P^{\infty})$ and the standard Atiyah-Hirzebruch spectral sequence may be invoked to show, following Dold,

$$E\mathbb{R}(n)^\star(\mathbb{C}P^\infty) \cong E\mathbb{R}(n)^\star[[u]]$$

and similarly for $c(E\mathbb{R}(n))$.

Consider the Real bundle $\gamma^{\otimes 2}$. The Real Chern class of this is $[2](u)$. Let $\widetilde{RP}^\infty$ denote the unit sphere bundle of $\gamma^{\otimes 2}$. $\widetilde{RP}^\infty$ may be identified with the space of real lines in $\mathbb{C}^\infty$ and it admits an involution given by complex conjugation. The real projective space RP^∞ (with trivial involution) sits inside $\widetilde{RP}^\infty$ as a non-equivariant deformation retract. We therefore have the isomorphism $c(E\mathbb{R}(n))^\star(RP^\infty) \simeq c(E\mathbb{R}(n))^\star(\widetilde{RP}^\infty)$. We can calculate the cohomology of $\widetilde{RP}^\infty$ using the Gysin sequence for the bundle $\gamma^{\otimes 2}$:

$$\ldots \to c(E\mathbb{R}(n))^{(k-1)+(l-1)\alpha}(\mathbb{C}P^\infty) \to c(E\mathbb{R}(n))^{k+l\alpha}(\mathbb{C}P^\infty) \to c(E\mathbb{R}(n))^{k+l\alpha}(\widetilde{RP}^\infty)$$

$$\to c(E\mathbb{R}(n))^{k+(l-1)\alpha}(\mathbb{C}P^\infty) \to \ldots$$

Since multiplication by the Chern class of $\gamma^{\otimes 2}$

$$[2](u) = 2u +_F v_1u^2 +_F \ldots +_F v_n u^{2^n} = 2u + \ldots + v_n u^{2^n} + \ldots \tag{9}$$

(the dots indicate terms with higher powers of u) is injective, we conclude from the Gysin sequence

$$c(E\mathbb{R}(n))^\star(RP^\infty) \cong c(E\mathbb{R}(n))^\star[[u]]/([2](u)).$$

Let $c(E\mathbb{R}(n))^*(RP^\infty)$ be the subring of elements in integral degree. Then

$$E\mathbb{R}(n)^*(RP^\infty) \cong ER(n)^*(RP^\infty)$$

since RP^∞ has trivial $\mathbb{Z}/2$ action and $ER(n) = c(E\mathbb{R}(n))^{\mathbb{Z}/2}$.

We wish to convert this result to a corresponding statement for $ER(n)$. We recall the invertible element $y(n)$ in degree $\lambda(n)+\alpha$ and use it to convert the Gysin sequence to an exact sequence with integer degrees.

$$\begin{tikzcd}
\cdots \arrow[r] & E\mathbb{R}(n)^{k-1-\alpha}(\mathbb{C}P^\infty) \arrow[r, "{[2](u)}"] & E\mathbb{R}(n)^{k}(\mathbb{C}P^\infty) \arrow[r] & E\mathbb{R}(n)^{k}(\widetilde{RP^\infty}) \arrow[r] & \cdots \\
& E\mathbb{R}(n)^{k-1+\lambda(n)}(\mathbb{C}P^\infty) \arrow[u, "\simeq", "y(n)"'] \arrow[ur] & & E\mathbb{R}(n)^{k}(RP^\infty) \arrow[u, "="] &
\end{tikzcd}$$

We replace u by $u' = uy(n)$, v_i by $v_i' = v_i y(n)^{-2^i+1}$, and in the formal group law $F(p,q) = p + q + \Sigma_{i,j} a_{i,j} p^i q^j$, replace $a_{i,j}$ by $a'_{i,j} = a_{i,j} y(n)^{-i-j+1}$ and $F(p,q)$ by $F'(p,q) = p + q + \Sigma_{i,j} a'_{i,j} p^i q^j$, to obtain the new [2]-series $[2]'(u') = F'(u',u')$.

Now we drop all the $'$. As in [12], we obtain the Gysin sequence of integer-graded groups

$$\cdots \longrightarrow ER(n)^*(\mathbb{C}P^\infty) \xrightarrow{[2](u)} ER(n)^*(\mathbb{C}P^\infty) \longrightarrow ER(n)^*(RP^\infty) \longrightarrow \cdots$$

Also, we work with *cohomology* degrees from now on, by changing all the signs from homology degrees, since we work in cohomology. So u now has degree $-\lambda(n)+1$. Therefore we have

$$ER(n)^*(RP^\infty) \cong ER(n)^*[[u]]/[2](u)$$

where $u \in ER(n)^{1-\lambda(n)}(RP^\infty)$ and $v_k \in ER(n)^{(\lambda(n)-1)(2^k-1)}$

5 The Bockstein spectral sequence

We have the stable cofibration

$$\Sigma^{\lambda(n)} ER(n) \xrightarrow{x} ER(n) \longrightarrow E(n)$$

where $x \in ER(n)^{-\lambda(n)}$ and $\lambda(n) = 2^{2n+1} - 2^{n+2} + 1$. The fibration gives us a long exact sequence

$$\begin{tikzcd}
ER(n)^*(X) \arrow[rr, "x"] & & ER(n)^*(X) \arrow[dl, "\rho"] \\
& E(n)^*(X) \arrow[ul, "\partial"] &
\end{tikzcd} \tag{10}$$

where x lowers the degree by $\lambda(n)$ and ∂ raises the degree by $\lambda(n)+1$. This leads to the Bockstein spectral sequence, which will completely determine $M = ER(n)^*(X)/(x)$ as a subring of $E(n)^*(X)$. We know that $x^{2^{n+1}-1} = 0$ so there can be only $2^{n+1} - 1$ differentials.

We filter M,

$$0 = M_0 \subset M_1 \subset M_2 \subset \ldots \subset M_{2^{n+1}-1} = M$$

by submodules

$$M_r = \operatorname{Ker}\left[x^r : \frac{ER(n)^*(X)}{x} \to \frac{x^r ER(n)^*(X)}{x^{r+1}}\right]$$

so that M_r/M_{r-1} gives the x^r-torsion elements of $ER(n)^*(X)$ that are non-zero in M.

We collect the basic facts about the spectral sequence in the following theorem in [12]. $E(n)$ is a complex oriented spectrum with a complex conjugation action. Denote this action by c.

Theorem 5.1. *In the Bockstein spectral sequence for* $ER(n)^*(X)$

1. *The exact couple (10) gives rise to a spectral sequence,* E^r, *of* $ER(n)^*$*-modules, starting with*

$$E^1 \simeq E(n)^*(X).$$

2. $E^{2^{n+1}} = 0$

3. $Im\, d^r \simeq M_r/M_{r-1}$.

4. *The degree of* d^r *is* $r\lambda(n)+1$.

5. $d^r(ab) = d^r(a)b + c(a)d^r(b)$

6. $d^1(z) = v_n^{-(2^n-1)}(1-c)(z)$ *where* $c(v_i) = -v_i$.

7. *If* $c(z) = z$ *in* E^1, *then* $d^1(z) = 0$. *If* $c(z) = z$ *in* E^r *then* $d^r(z^2) = 0$.

8. *The following are all vector spaces over* $\mathbb{Z}/2$*:*

$$M_j/M_i,\ (j \geq i > 0)\ \textit{and}\, E^r, (r \geq 2).$$

Proof Most of the results follow from the basic properties of an exact couple and the fact that $x^{2^{n+1}-1} = 0$. We have the complex conjugation for our involution on $E(n)^*(X)$ and trivial involution on $ER(n)^*(X)$. Assuming the formula for d^1 we obtain the product formula for d^1:

$$d^1(ab) = v_n^{-(2^n-1)}(1-c)(ab) = v_n^{-(2^n-1)}(ab - c(ab)) = v_n^{-(2^n-1)}(ab - c(a)c(b))$$

$$= v_n^{-(2^n-1)}((a-c(a))b + c(a)(b-c(b))) = d^1(a)b + c(a)d^1(b).$$

The product rule for d^r follows from that of d^1. If $c(z) = z$:

$$d^1(z) = v_n^{-(2^n-1)}(1-c)(z) = v_n^{-(2^n-1)}(z-z) = 0.$$

For the second case $d^1(z^2) = 0$ because z^2 is invariant under c. For $r > 1$,

$$d^r(z^2) = zd^r(z) + c(z)d^r(z) = 2zd^r(z)$$

which is zero since we are working modulo 2 for $r > 1$.

Let us continue to assume the formula for d^1 and show the $\mathbb{Z}/2$ vector spaces. It is enough to show this for E^2. Start with an arbitrary d^1-cycle $y \in E^1$ such that

$2y \neq 0$. Observe that

$$d^1(v_n^{2^n-1}) = v_n^{-(2^n-1)}(1-c)(v_n^{2^n-1}) = v_n^{-(2^n-1)}(v_n^{2^n-1} + v_n^{2^n-1}) = 2.$$

This shows that $2x = 0$. Consider the element $v_n^{2^n-1}y$.

$$d^1(v_n^{2^n-1}y) = yd^1(v_n^{2^n+1}) + c(v_n^{2^n-1})d^1(y) = 2y + 0.$$

Therefore no multiplication by 2 survives to E^2.

This is all we need to show that the M_j/M_i for $j \geq i > 0$ are $\mathbb{Z}/2$ vector spaces.

Finally the formula for the differential d^1 follows from Propsition 3.7.

Note that the image of $ER(n)^*(X) \to E(n)^*(X)$ consists of targets of the differentials and therefore always have the differentials trivial on them. Also anything in the image is trivial under the action of c.

Since $ER(n)^*(-)$ is $2^{n+2}(2^n-1)$-periodic we will consider it as graded over $\mathbb{Z}/(2^{n+2}(2^n-1))$. We have to do the same then for $E(n)^*(-)$. We can do this by setting the unit $v_n^{2^{n+1}} = 1$ in the homotopy of $E(n)$. (This does not lose any information since we can always recover the original by inserting powers of $v_n^{2^{n+1}}$ to make the degrees match.)

5.1 The spectral sequence for $ER(2)^*$

For the Bockstein spectral sequence of Theorem 5.1 to be useful, we need to know the ring $ER(n)^*$. From now on we concentrate on the case $n = 2$. This spectral sequence begins with $E^1 = E(2)^*$ which is just a free $\mathbb{Z}_{(2)}[v_1]$-module on a basis given by v_2^i for $0 \leq i < 8$. We are grading mod 48. Since all elements of E^1 have even degree and degree $\deg d^r = 17r + 1$, $d^r = 0$ for n even. As $E^8 = 0$, we only have d^1, d^3, d^5, d^7 to consider.

We have the differential d^1 acting as follows:

$$d^1(v_2^{2s+1}) = v_2^{-3}(1-c)v_2^{2s+1} = v_2^{-3}2v_2^{2s+1} = 2v_2^{2s-2}.$$

Similarly $d^1(v_2^{2s}) = 0$.

However multiplication by v_1 doesn't behave well with respect to this differential. The problem is that $v_1 \in E(2)^*$ does not lift to $ER(2)^*$. We will need a substitute for v_1. We shall use the element $\alpha = v_1^{ER(2)} = y(2)^{-1}v_1 \in ER(2)^{\lambda(2)-1}$. The image of α is $v_1 v_2^5 \in E(2)^{-32}$. Because v_2 is a unit this a good substitute for ordinary v_1. Furthermore this is invariant under c because it is in the image of the map from $ER(2)^*$ and is a permanent cycle. Or we could just see ths by observing that there are an even number of v's. We rewrite the homotopy of $E(2)$ as $\mathbb{Z}_{(2)}[\alpha, v_2^{\pm 1}]$ but again set $v_2^8 = 1$.

Now back to the computation of d^1 on v_2^{2s+1} where the E_1 term is a free $\mathbb{Z}_{(2)}[\alpha]$-module on generators v_2^i, $0 \leq i < 8$.

$$d^1(\alpha^k v_2^{2s+1}) = d^1(\alpha^k)v_2^{2s+1} + c(\alpha^k)d^1(v_2^{2s+1}) = 0 + \alpha^k 2v_2^{2s-2}.$$

But this really follows from the fact that we have a spectral sequence of $ER(2)^*$-modules. Thus the d^1-cycles form a free $\mathbb{Z}_{(2)}[\alpha]$-module generated by $\{1, v_2^2, v_2^4, v_2^6\}$ and the d^1-boundaries form the free submodule with basis $\{\alpha_0, \alpha_1, \alpha_2, \alpha_3\}$, where $\alpha_i = 2v_2^{2i}$. In particular, $\alpha_0 = 2$. Thus $E^2 = E^3$ is the free $\mathbb{F}_2[\alpha]$-module with the basis (the images of) $\{1, v_2^2, v_2^4, v_2^6\}$.

By [12], $d^3(v_2^2) = \alpha v_2^4$. Since d^3 is a derivation, $d^3(v_2^6) = \alpha$, and the only elements of E^4 are 1 and v_2^4. Since $\deg d^5 = 38$, $d^5 = 0$. We must have $d^7(v_2^4) = 1$ to make $E^8 = 0$.

We can read off M as a $\mathbb{Z}_{(2)}[\alpha]$-submodule of E^1. $M_1 = \mathrm{Im} d^1$ is the free module on basis $\{\alpha_0, \alpha_1, \alpha_2, \alpha_3\}$. M_3 is generated as a module by adding the elements α and

$w = \alpha v_2^4$, which make M_3/M_1 the free $\mathbb{F}_2[\alpha]$-module with basis $\{\alpha, w\}$. Finally, the only new element of M_7 is 1. The rest of the module structure is given by

$$2\alpha = \alpha\alpha_0, 2w = \alpha\alpha_2, 2.1 = \alpha_0, \alpha.1 = \alpha \tag{11}$$

Further, M is a subring of E^1, generated by $\alpha_1, \alpha_2, \alpha_3, \alpha$ and w. The products not already given are

$$\begin{aligned} \alpha_s\alpha_t &= 2\alpha_{s+t} \;\; (\text{taking } s+t \bmod 4) && (12)\\ w\alpha_s &= \alpha\alpha_{s+2} \;\; (\text{taking } s+2 \bmod 4) && (13)\\ w^2 &= \alpha^2 && (14) \end{aligned}$$

To obtain $ER(2)^*$, we must unfilter $M = ER(2)^*/(x)$ and add the generator x. We lift each generator α_s and w to $ER(2)^*$, keeping the same names, with the relations

$$\alpha_s x = 0, \alpha x^3 = 0, w x^3 = 0, x^7 = 0. \tag{15}$$

By sparseness, each α_s and w lifts *uniquely* to $ER(2)^*$; further, the module actions (11) and multiplications (12), (13), (14) also lift uniquely and hold in $ER(2)^*$, not merely mod(x).

Proposition 5.1. *$ER(2)^*$ is graded over $\mathbb{Z}/48$. It is generated as a ring by elements,*

$$x, w, \alpha, \alpha_1, \alpha_2, \alpha_3$$

of degrees -17,-8,-32,-12,-24 and -36 respectively, with relations and products as listed above.

5.2 The Bockstein Spectral Sequence for $ER(2)^*(RP^\infty)$

As always, we have the split short exact sequence

$$0 \to ER(2)^*(X,*) \to ER(2)^*(X) \to ER(2)^*(*) \to 0$$

for any space X with baspoint $*$. Now that we know $ER(2)^*(*) = ER(2)^*$, we will concentrate on $ER(2)^*(X,*)$. Nevertheless, we need to use the action of $ER(2)^*$ on $ER(2)^*(X)$ and hence $ER(2)^*(X,*)$; furthermore, this action extends to actions of the Bockstein spectral sequences.

The $E(2)^*$-cohomology of RP^∞ can be computed from the Gysin sequence

$$\dots E(2)^{k-2}(\mathbb{C}P^\infty) \xrightarrow{[2](x_2)} E(2)^k(\mathbb{C}P^\infty) \longrightarrow E(2)^k(\widetilde{RP^\infty}) \longrightarrow E(2)^{k-1}(\mathbb{C}P^\infty) \dots$$

where $E(2)^*(\mathbb{C}P^\infty) \simeq E(2)^*[[x_2]]$, $x_2 \in E(2)^2(\mathbb{C}P^\infty)$. From above $E(2)^*(RP^\infty) \simeq E(2)^*[[x_2]]/([2](x_2))$. Recall that $ER(2)^*(RP^\infty) = ER(2)^*[[u]]/([2](u))$ where we have $u \in ER(2)^{1-\lambda(2)}(RP^\infty)$. We will replace the x_2 by the image of $u \in ER(2)^{-16}(RP^\infty)$ which we also call $u \in E(2)^{-16}(RP^\infty)$, which is really $v_2^3 x_2$. Likewise we replace the usual $v_1 \in E(2)^{-2}$ with $v_2^5 v_1 = \alpha \in E(2)^{-32}$ which comes from $\alpha \in ER(2)^{-32}$. The element $w \in ER(2)^{-8}$ maps to $\alpha v_2^4 = v_2 v_1 \in E(2)^{-8}$. These changes are necessary because x_2 and v_1 are not in the image of $ER(2)$-cohomology.

We will describe our groups in terms of a *2-adic basis* in the sense of [12], i.e, a set of elements such that any element in our group can be written as a unique sum of these elements with coefficients 0 or 1 (where the sum is allowed to be a formal power series in u). In the ring $E(2)^*(RP^\infty)$ we have $2u = \alpha u^2 + \dots$, therefore the 2-adic basis is given by $v_2^i \alpha^k u^j$, $(0 \leq i < 8, 0 \leq k, 1 \leq j)$.

The original relation $[2](u) = 2u +_F v_1 u^2 +_F v_2 u^4$ for $E\mathbb{R}(2)$ converts to the relation

$$[2](u) = 2u +_F \alpha u^2 +_F u^4 \tag{16}$$

since v_1 is replaced by $v_1(v_2^3)^{-1} = v_1 v_2^5 = \alpha$ and v_2 is replaced by $v_2(v_2^3)^{-3} = v_2^{-8} = 1$. Because $2x = 0$, x times the relation (16) gives us $0 = x(\alpha u^2 +_F u^4)$. Therefore from the point of view of x^1-torsion αu^2 can be replaced with $u^4 + \ldots$. Similarly, if we multiply by x^3 and use the relation $x^3\alpha = 0$ we end up with $x^3 u^4 = 0$.

Since $u \in E(2)^*(RP^\infty)$ is in the image from $ER(2)^*(RP^\infty)$ our differentials commute with multiplication by u and also commute with multiplication by α. The d^1 differential creates a relation coming from our relation $0 = 2u +_F \alpha u^2 +_F u^4$ when $2u$ is set to zero. So in E^2, we have $\alpha u^2 \equiv u^4 + \ldots$. The Bockstein spectral sequence goes like this:

Theorem 5.2. *$E^1 = E(2)^*(RP^\infty, *)$ is represented by*

$$v_2^i \alpha^k u^j \quad (0 \le i \le 7, \quad 0 \le k, \quad 1 \le j).$$

$$d^1(v_2^{2s-5}\alpha^k u^j) = 2v_2^{2s}\alpha^k u^j = v_2^{2s}\alpha^{k+1}u^{j+1} + \ldots$$

$E^2 = E^3$ is given by:

$$v_2^{2s}\alpha^k u, \quad v_2^{2s}u^j \quad (2 \le j, \quad 0 \le s \le 4, \quad 0 \le k)$$

$$d^3(v_2^{4s-2}\alpha^k u) = v_2^{4s}\alpha^{k+1}u, \;\; d^3(v_2^{4s-2}u^j) = v_2^{4s}\alpha u^j = v_2^{4s}u^{j+2} + \ldots$$

$E^4 = E^5 = E^6 = E^7$ is given by:

$$v_2^4 u^{\{1-3\}}, \quad u^{\{1-3\}}$$

$$d^7(v_2^4 u^{\{1-3\}}) = u^{\{1-3\}}.$$

The x^1-torsion generators are given by:

$$\alpha_i \alpha^k u^j \quad (0 \leq i, \quad 0 \leq k, \quad 1 \leq j)$$

where $\alpha_0 = 2$.

The x^3-torsion generators are given by:

$$\alpha^{k+1}u,\ \alpha^k wu\ (k \geq 0),\ u^j\ (j \geq 4),\ wu^j\ (j \geq 2).$$

The only x^7-torsion generators are

$$u^{\{1-3\}}.$$

In degrees that are multiples of 8 (denoted $8*$), the description of $ER(2)^*(RP^\infty, *)$ simplifies enormously. As x is the only generator whose degree is not a multiple of 4, and x^4 kills everything except powers of u, multiplication by powers of x produces no new elements in degree $8*$.

Corollary 5.1. *The homomorphism*

$$ER(2)^{8*}(RP^\infty, *) \to E(2)^{8*}(RP^\infty, *)$$

is injective, and is almost surjective – the only elements not hit are $v_2^4 u^j$ for $1 \leq j \leq 3$.

We shall compute the Bockstein spectral sequence associated to $ER(2)$ for the odd-dimensional projective space RP^{16K+9}. This will give us new non-immersion results from [13]. We will have to introduce some new elements for the computations in the next section. The Atiyah-Hirzebruch spectral sequence for $ER(2)^*(RP^2)$ gives elements x_1 and x_2 in filtration degrees 1 and 2. As a $ER(2)^*$ module $ER(2)^*(RP^2)$

is generated by elements we will call z_{-16} represented by xx_1 and z_2 represented by x_2. $z_{-16} = u \in ER(2)^*(RP^2)$.

For the cofibration $S^1 \to RP^2 \to S^2$, the long exact sequence

$$\begin{tikzcd}
ER(2)^*(S^1) \arrow[dr, "\partial"'] & & ER(2)^*(RP^2) \arrow[ll, "i^*"'] \\
& ER(2)^*(S^2,*) \arrow[ur, "\rho^*"'] &
\end{tikzcd}$$

is given by $\partial(i_1) = 2i_2, \rho^*(i_2) = z_2$, and $i^*(u) = xi_1$.

We know that $\Sigma^{2n-2}ER(2)^*(RP^2,*) \simeq ER(2)^*(RP^{2n}/RP^{2n-2},*)$. We have elements $z_{2n-18}, z_{2n} \in ER(2)^*(RP^{2n}/RP^{2n-2},*)$. These elements map to $v_2^{5n+3}u^n$ and $v_2^{5n}u^n$ respectively in $E(2)^*(RP^{2n}/RP^{2n-2},*)$ by [12].

6 $ER(2)^*(RP^{16K+9},*)$

In [12], Kitchloo and Wilson considered only even dimensional real projective spaces. Our object is to extend the results to certain odd-dimensional cases.

Proposition 6.1. *The element* $u^{8K+5} \in ER(2)^*(RP^{16K+10})$ *maps to a non-zero element of* $ER(2)^*(RP^{16K+9})$.

Note that this cannot happen for a complex oriented cohomology theory.

Proof We use the exact sequence

$$ER(2)^*(S^{16K+10},*) \xrightarrow{q^*} ER(2)^*(RP^{16K+10}) \longrightarrow ER(2)^*(RP^{16K+9})$$

We only have to show that u^{8K+5} is not in the image of q^*. Now $ER(2)^*(S^{16K+10},*)$ is the free $ER(2)^*$-module generated by the element i_{16K+10}, and q^* is known.

The structure of $ER(2)^*(RP^{16K+10})$ is given by Theorems 13.2 and 13.3 of [12].

We give a complete description of M, where

$$ER(2)^*(RP^{16K+10},*)/xER(2)^*(RP^{16K+10},*) \simeq M \subset E(2)^*(RP^{16K+10},*)$$

as a submodule of $E(2)^*(RP^{16K+10}) = \mathbb{Z}_{(2)}[\alpha, v_2^{\pm 1}][u]/(u^{8K+6}, [2](u))$. We describe M by specifying a 2-adic basis. As $d^r = 0$ for r even, M is filtered by $0 = M_0 \subset M_1 = M_2 \subset M_3 = M_4 \subset M_5 = M_6 \subset M_7 = M$, where $M_r/M_{r-1} \simeq \mathrm{Im} d^r$ and d^r is the differential in the Bockstein spectral sequence. Elements of M_r not in M_{r-1} lift to x^r-torsion elements of $ER(2)^*(RP^{16K+10},*)$.

As both α and u come from $ER(2)$-cohomology, we may describe M from section 5 as a filtered $\mathbb{Z}_2[\alpha, u]$-module. We write z_t for various elements of $ER(2)^*(RP^{16K+10},*)$ and $\bar{z}_t$ for it's image in M, where t denotes the degree.

$\alpha^k u^j (u\alpha_s) \in M_1$ for $k \geq 0$, $0 \leq j \leq 8K+4$, $0 \leq s \leq 3$, where $u\alpha_s = 2v_2^{2s}u = d^1(v_2^{2s+3}u)$. Note that $\alpha_0 = 2$.

$\alpha^k(u\alpha) \in M_3$ for $k \geq 0$, where $u\alpha = d^3(v_2^{-2}u)$.

$\alpha^k(uw) \in M_3$ for $k \geq 0$, where $uw = d^3(v_2^2 u) = uv_2^4\alpha$.

$\alpha^k u^j \beta_0 \in M_3$ for $k \geq 0$, $0 \leq j \leq 8K+1$, where $\beta_0 = d^3(v_2^{-2}u^2) = \alpha u^2 \equiv u^4 + \dots$ mod $2 = \alpha_0$.

$\alpha^k u^j \beta_1 \in M_3$ for $k \geq 0$, $0 \leq j \leq 8K+1$ where $\beta_1 = d^3(v_2^2 u^2) = wu^2$.

$\alpha^k \gamma_0 \in M_3$ for $k \geq 0$, where $\gamma_0 = v_2\alpha u^{8K+5} = d^3(v_2^{-1}u^{8K+5})$.

$\alpha^k \gamma_1 \in M_3$ for $k \geq 0$, where $\gamma_1 = v_2^5\alpha u^{8K+5} = d^3(v_2^3 u^{8K+5})$.

$\bar{z}_{16K+10} \in M_5$, where $z_{16K+10} = q^* i_{16K+10}$ is induced from S^{16K+10}. Then $\bar{z}_{16K+10} = v_2 u^{8K+5} = d^5(v_2^2 u^{8K+4})$. Note that $\alpha\bar{z}_{16K+10} = \gamma_0$ and $w\bar{z}_{16K+10} = \gamma_1$.

$\bar{z}_{16K-14} \in M_5$, where $\bar{z}_{16K-14} = v_2^5 u^{8K+5} = d^5(v_2^6 u^{8k+4})$. Note that $\alpha\bar{z}_{16K-14} = \gamma_1$ and $w\bar{z}_{16K-14} = \gamma_0$.

$\bar{z}_{16K+4} \in M_7$, where $\bar{z}_{16K+4} = v_2^2 u^{8K+5} = d^7(v_2^6 u^{8K+5})$.

$u^j \in M_7$, for $1 \leq j \leq 3$, since $d^7(v_2^4 u^j) = u^j$.

The action of α on M is clear, except for $\alpha\bar{z}_{16K+4} = -u^{8K+4}\alpha_1$. The relations involving u can also be determined, but are not useful. We really need the corresponding lifted relations in $ER(2)$-cohomology, where they hold only mod(x). Again, we quote

$$u^{8K+6} = x^2 z_{16K-14},\ u^{8K+7} = x^4 z_{16K+4},\ u^{8K+8} = 0. \tag{17}$$

Now in $ER(2)$-cohomology, $q^* i_{16K+10} = v_2 u^{8K+5}$, from (13.1) [12]. Since the degrees of α_2, α, w and x are $-12s, 16, 40$ and -17 respectively, mod 48, the only elements in $ER(2)^* i_{16K+10}$ of degree $-16(8K+5)$ have the form $\alpha^k w x^2 i_{16K+10}$. Then we have $q^*(\alpha^q w x^2 i_{16K+10}) = v_2 \alpha^q w x^2 u^{8K+5}$, which lies in the ideal (x) and is not the same as u^{8K+5}.

6.1 The Bockstein spectral sequence for $ER(2)^*(RP^{16K+9})$

We compute this cohomology by sandwiching it between the $ER(2)$-cohomology of RP^{16K+10} and RP^{16K+8}, which we know, in the commutative diagram of exact sequences

$$\begin{tikzcd}
ER(2)^*(S^{16K+10},*) \arrow[r, "="] \arrow[d, "\rho^*"] & ER(2)^*(S^{16K+10},*) \arrow[d] & \\
ER(2)^*(\frac{RP^{16K+10}}{RP^{16K+8}},*) \arrow[r] \arrow[d, "i^*"] & ER(2)^*(RP^{16K+10},*) \arrow[r] \arrow[d] & ER(2)^*(RP^{16K+8},*) \arrow[d, "="] \\
ER(2)^*(S^{16K+9},*) \arrow[r] \arrow[d, "2"] & ER(2)^*(RP^{16K+9},*) \arrow[r] & ER(2)^*(RP^{16K+8},*) \\
ER(2)^*(S^{16K+9},*) & &
\end{tikzcd} \tag{18}$$

The E^1-term for the Bockstein spectral sequence is just $E(2)^*(RP^{16K+9},*)$, which decomposes [4] as

$$E(2)^*(RP^{16K+8},*) \oplus E(2)^*(S^{16K+9},*).$$

via the maps

$$RP^{16K+8} \to RP^{16K+9} \to S^{16K+9}$$

(In general, $E(2)^*(RP^{2n}) = E(2)^*[u]/(u^{n+1},[2](u))$, as $E(2)$ is complex oriented and $E(2)^*$ has no 2-torsion.) The even part of the E^1-term has the 2-adic basis

$$v_2^i \alpha^k u^j \quad (0 \leq i \leq 7, \quad 0 \leq k, \quad 0 \leq j \leq 8K+4)$$

The odd part is a free $\mathbb{Z}_{(2)}$-module with basis

$$v_2^i \alpha^k i_{16K+9} \quad (0 \leq i \leq 7, \quad 0 \leq q, \quad 0 \leq k)$$

Since d^1 is of even degree we can just read off our d^1 from Theorem 13.2 [12] for RP^{16K+8} and section 5 of [12] for the S^{16K+9} part.

$$d^1(v_2^{2s-5}\alpha^k u^j) = 2v_2^{2s}\alpha^k u^j = v_2^{2s}\alpha^{k+1}u^{j+1} + \dots \quad (j \leq 8K+4)$$

$$d^1(v_2^{2s+1}\alpha^k i_{16K+9}) = 2v_2^{2s-2}\alpha^k i_{16K+9}$$

Thus E^2 is given by

$$v_2^{2s}\alpha^k u \quad (k \geq 0), \quad v_2^{2s}u^j \quad (1 \leq j \leq 8K+4), \quad v_2^{2s+1}\alpha^k u^{8K+4} \quad (0 \leq k),$$

$$v_2^{2s}\alpha^k i_{16K+9}$$

d^2 has odd degree 35. Since we have both odd and even degree elements in the E^2-term, d^2 might very well be non-trivial. If it is, then by naturality, it must have its source in the RP^{16K+8} part and target in the S^{16K+9} part. Also, the source cannot be anything from the E^2-term for the BSS for RP^∞, for we know that d^2 is trivial there. Therefore the only possible sources are $v_2^{2s+1}\alpha^k u^{8K+4}$ with possible targets $v_2^{2s}\alpha^k i_{16K+9}$.

Now, since v_2^2 is a unit, if there is a d^2, it must be non-zero on $v_2^{-1}u^{8K+4}$ which has degree $+6-16(8K+4)$ which is $-10+16K$ mod 48. The degree of the target must be this plus 35. The possible targets have degrees $-12s-32k+16K+9$. The only solutions are αi_{16K+9}, $\alpha^4 i_{16K+9}$ etc.

If d^2 is non-zero our guess would be $d^2(v_2^{-1}u^{8K+4}) = \alpha i_{16K+9}$. Thus we need to show that $x^2\alpha i_{16K+9} = 0$ in order for our guess to be correct.

The left column of (18) shows that the $ER(2)^*$-module $ER(2)^*(RP^{16K+10}/RP^{16K+8}, *)$ is generated by two elements $z_{16K+10} = \rho^* i_{16K+10}$ and z_{16K-8}, where $i^* z_{16K-8} = x i^{16K+9}$.

We want to show that $x^2\alpha i_{16K+9} = 0$ in $ER(2)^*(RP^{16K+9}, *)$. This is the image of the same named element in $ER(2)^*(S^{16K+9}, *)$ which lifts to $x\alpha z_{16K-8} \in ER(2)^*(\frac{RP^{16K+10}}{RP^{16K+8}})$. This maps to $x\alpha v_2^4 u^{8K+5}$ in $ER(2)^*(RP^{16K+10}, *)$ from (13.1) of [12]. Since $2x = 0$, we can use the relation $[2](u) = 2u +_F \alpha u^2 +_F u^4 = 0$ on αv_2^2, and we get

$$x\alpha v_2^4 u^{8K+5} = x v_2^4 u^{8K+7} + \dots$$

The least power of u which is zero in $ER(2)^*(RP^{16K+10})$ is $8K+8$. We have noted that $u^{8K+7} = x^4 z_{16K+4}$, so that we have $x^5 v_2^4 z_{16K+4}$, which is non-zero and does not lift to S^{16K+10}. It follows that $x^2\alpha i_{16K+9} \neq 0$.

However, if we multiply the whole calculation by α^3, we get $\alpha^3 x^5 v_2^4 z_{16K+4} = 0$, as $x^3\alpha = 0$. So $x^2\alpha^4 i_{16K+9} = 0$, and we conclude that

$$d^2(v_2^{2s+1}\alpha^k u^{8K+4}) = v_2^{2s+2}\alpha^{k+4} i_{16K+9}$$

Then, E^3 is given by

$$v_2^{2s}\alpha^k u \quad (0 \leq k), \;\; v_2^{2s}u^j \quad (1 \leq j \leq 8K+4), \;\; v_2^{2s}\alpha^{\{0-3\}}i_{16K+9}.$$

d^3 is even degree so the even and odd parts don't mix under the differential. On both parts we already know the d^3 differential:

$$d^3(v_2^{\{6,2\}}\alpha^k u) = v_2^{\{0,4\}}\alpha^{k+1}u$$

$$d^3(v_2^{\{6,2\}}u^j) = v_2^{\{0,4\}}\alpha u^j = v_2^{\{0,4\}}u^{j+2} \quad (1 \leq j \leq 8K+2)$$

$$d^3(v_2^{4s-2}\alpha^{\{0-2\}}i_{16K+9}) = v_2^{4s}\alpha^{\{1-3\}}i_{16K+9}$$

$$d^3(v_2^{4s}\alpha^{\{0-3\}}i_{16K+9}) = 0$$

Thus E^4 is given by

$$v_2^{\{0,4\}}u^{\{1-3\}}, \quad v_2^{\{6,2\}}u^{8K+3,8K+4}, \quad v_2^{4s}i_{16K+9}.$$

d^4 has degree 21, which is odd. So it must go from the RP^{16K+8} part to the S^{16K+9} part. d^4 must be zero on anything in the image from RP^∞. So our non-zero differentials have possible sources $v_2^{\{6,2\}}u^{\{8K+3,8K+4\}}$, and possible targets $v_2^{4s}i_{16K+9}$ and $v_2^{\{6,2\}}\alpha^3 i_{16K+9}$. Let's compute the degrees (mod 48).

$$v_2^6u^{8K+3} : -36 - 16(8K+3) \equiv -36 + 16K \longrightarrow -15 + 16K$$

$$v_2^2u^{8K+3} : -12 - 16(8K+3) \equiv -12 + 16K \longrightarrow 9 + 16K$$

$$v_2^6u^{8K+4} : -36 - 16(8K+4) \equiv -4 + 16K \longrightarrow 17 + 16K$$

$$v_2^2u^{8K+4} : -12 - 16(8K+4) \equiv 20 + 16K \longrightarrow 41 + 16K$$

Comparing with degrees of the possible targets we see that the only possible differentials are:

$$d^4(v_2^{\{6,2\}}u^{8K+3}) = v_2^{\{0,4\}}i_{16K+9}$$

This must be true if i_{16K+9} is x^4-torsion. We invoke the commutative diagram (18) used before. Consider x^3z_{16K-8} in $ER(2)^*(\frac{RP^{16K+10}}{RP^{16K+8}})$. The following diagram shows its images in the lower left hand square of the commutative diagram.

$$\begin{tikzcd}
x^3z_{16K-8} \arrow[r, mapsto] \arrow[d, mapsto] & x^3v_2^4u^{8K+5} \arrow[d, mapsto] \\
x^4i_{16K+9} \arrow[r, mapsto] & x^4i_{16K+9}
\end{tikzcd}$$

The element in the upper right-hand corner is zero since $x^3u^4 = 0$. This shows i_{16K+9} is indeed x^4-torsion. We obtain our E^5-term

$$v_2^{\{0,4\}}u^{\{1-3\}}, \quad v_2^{\{6,2\}}u^{8K+4}.$$

d^5 has even degree. For dimensional reasons the differentials must be zero in the odd part, and Theorem 13.2 [12] determines that it is zero for the even part.

d^6 has degree 7. Again, it must go from even part to odd part by naturality. By sparseness, $d^6 = 0$.

Since d^7 has even degree, the differential does not mix odd and even degrees. First of all in the even part we have

$$d^7(v_2^4u^{\{1-3\}}) = u^{\{1-3\}}$$

Also by mapping to RP^{16K+8}(page 23 [12]) we get that

$$d^7(v_2^6u^{8K+4}) = v_2^2u^{8K+4}$$

We collect our results in the following theorem.

Theorem 6.1. *The Bockstein spectral sequence for $ER^*(RP^{16K+9}, *)$ is as follows:*

E^1

$$v_2^i \alpha^k u^j \quad (0 \le i \le 7, \quad 0 \le k, \quad 0 \le j \le 8K+4)$$

$$v_2^i \alpha^k i_{16K+9} \quad (0 \le i \le 7, \quad 0 \le k)$$

$$d^1(v_2^{2s-5}\alpha^k u^j) = 2v_2^{2s}\alpha^k u^j = v_2^{2s}\alpha^{k+1}u^{j+1} + \dots \quad (j \le 8K+3)$$

$$d^1(v_2^{2s+1}\alpha^k i_{16K+9}) = 2v_2^{2s-2}\alpha^k i_{16K+9}$$

where $v_2^i \alpha^k i_{16K+9}$ generates a free $\mathbb{Z}_{(2)}$-module in E^1.

E^2

$$v_2^{2s}\alpha^k u \quad (k \ge 0), \quad v_2^{2s}u^j \quad (1 \le j \le 8K+4), \quad v_2^{2s+1}\alpha^k u^{8K+4} \quad (0 \le k),$$

$$v_2^{2s}\alpha^k i_{16K+9}$$

$$d^2(v_2^{2s+1}\alpha^k u^{8K+4}) = v_2^{2s+2}\alpha^{k+4} i_{16K+9}$$

E^3

$$v_2^{2s}\alpha^k u \quad (0 \le k), \ \ v_2^{2s}u^j \quad (1 \le j \le 8K+4), \ \ v_2^{2s}\alpha^{0-3} i_{16K+9}$$

$$d^3(v_2^{\{6,2\}}\alpha^k u) = v_2^{\{0,4\}}\alpha^{k+1}u$$

$$d^3(v_2^{\{6,2\}}u^j) = v_2^{\{0,4\}}\alpha u^j = v_2^{\{0,4\}}u^{j+2} \quad (2 \le j \le 8K+2)$$

$$d^3(v_2^{4s-2}\alpha^{\{0-2\}} i_{16K+9}) = v_2^{4s}\alpha^{\{1-3\}} i_{16K+9}$$

$$d^3(v_2^{4s}\alpha^{\{0-3\}} i_{16K+9}) = 0$$

E^4

$$v_2^{\{0,4\}}u^{\{1-3\}}, \quad v_2^{\{6,2\}}u^{\{8K+3,8K+4\}}, \quad v_2^{4s}i_{16K+9}$$

$$d^4(v_2^{\{6,2\}}u^{8K+3}) = v_2^{\{0,4\}}i_{16K+9}$$

$E^5 = E^6 = E^7$

$$v_2^{\{0,4\}}u^{\{1-3\}}, \ v_2^{\{6,2\}}u^{8K+4}, \ v_2^{\{6,2\}}\alpha^3 i_{16K+9}$$

$$d^7(v_2^4 u^{\{1-3\}}) = u^{\{1-3\}}, \quad d^7(v_2^6 u^{8K+4}) = v_2^2 u^{8K+4}$$

Next we identify all the elements in degree 8*.

Theorem 6.2. *A 2-adic basis of $ER(2)^{8*}(RP^{16K+9}, *)$ is given by the elements*

$$\alpha^k u^j, \ (k \geq 0, 1 \leq j \leq 8K+4)$$

$$v_2^4 \alpha^k u^j, \ (k \geq 1, 1 \leq j \leq 8K+4)$$

$$v_2^4 u^j, \ (4 \leq j \leq 8K+4)$$

$$x\alpha^k i_{16K+9}, \ xv_2^4\alpha^k i_{16K+9}, \ (k \geq 0)$$

Proof The first classes of elements represent the images of differentials in the spectral sequence that do not involve i_{16K+9}. As in [13], multiplication by powers of x leads to no new elements in degree $8*$. Those images involving i_{16K+9} provide x^2,x^3, or x^3-torsion, which may be multiplied by x.

Corollary 6.1. *There is an algebraic map*

$$ER(2)^{8*}(RP^{16K+9}) \to E(2)^{8*}(RP^{16K+10})$$

which only misses the elements $v_2^4 u^{\{1-3\}}$.

7 Non-Immersions

If RP^b immerses in $\mathbb{R}^c$, James showed [10] that there is an axial map

$$m : RP^b \times RP^{2^L-c-2} \to RP^{2^L-b-2}$$

for large L (meaning a map that is non-trivial on both axes). Specifically, to show that RP^{2n} does not immerse in $\mathbb{R}^{2k+1}$ we need to prove there is no axial map

$$m : RP^{2n} \times RP^{2^L-2k-3} \to RP^{2^L-2n-2} \tag{19}$$

Our strategy is to consider the class $u \in ER(2)^*(RP^{2^L-2n-2})$, which satisfies $u^{2^{L-1}-n} = 0$ when $n \equiv 0$ or 7 mod 8 (Theorem 1.6.[12]). We shall see that m^*u is known, in principle. If we can show that $(m^*u)^{2^{L-1}-n} \neq 0$, we have a contradiction.

Davis [4] used this approach, by using the complex-oriented cohomology theory $E(2)$ to deduce that RP^{2n} does not immerse in $\mathbb{R}^{2k}$ by showing there is no axial map

$$m : RP^{2n} \times RP^{2^L-2k-2} \to RP^{2^L-2n-2}$$

when $n = m + \alpha(m) - 1$ and $k = 2m - \alpha(m)$ for some m, where $\alpha(m)$ denotes the number of 1's in the binary expansion of m. We wish to improve this result to show that for certain n and k, (19) does not exist.

There is an axial map $m : RP^\infty \times RP^\infty \to RP^\infty$, which is the restriction of the map $\mathbb{C}P^\infty \times \mathbb{C}P^\infty \to \mathbb{C}P^\infty$ induced by the tensor product of the canonical Real line bundles. Therefore $m^*u = u_1 +_F u_2$, where u_1, u_2 and u are the Chern classes for the three copies of RP^∞.

If $m : RP^b \times RP^c \to RP^d$ is an axial map, the diagram

$$\begin{tikzcd}
RP^b \times RP^c \arrow[r, "m"] \arrow[d, "\subset"] & RP^d \arrow[d, "\subset"] \\
RP^\infty \times RP^\infty \arrow[r, "m"] & RP^\infty
\end{tikzcd}$$

commutes, as all axial maps $RP^b \times RP^c \to RP^\infty$ are homotopic. It follows that the same formula $m^* u = u_1 +_F u_2$ holds for this m. As the formal group law F is a formal power series in u_1 and u_2 over $\mathbb{Z}_{(2)}[\alpha]$ and $\deg u = -16$ and $\deg \alpha = 16$, we are interested only in degrees that are multiples of 16. This simplifies our work, as $ER(2)^{16*}(RP^\infty) \to E(2)^{16*}(RP^\infty)$ is a an isomorphism by [12].

We assume $k = 2 \bmod 8$, so that $2^L - 2k - 2 = 16K + 10$ and we can use Theorem 6.2. Consider the diagram

$$\begin{tikzcd}
ER(2)^{16*}(RP^\infty \times RP^\infty) \arrow[r] \arrow[d] & E(2)^{16*}(RP^\infty \times RP^\infty) \arrow[d] \\
ER(2)^{16*}(RP^{2n} \times RP^{16K+10}) \arrow[r] \arrow[d] & E(2)^{16*}(RP^{2n} \times RP^{16K+10}) \arrow[d] \\
ER(2)^{16*}(RP^{2n} \times RP^{16K+9}) \arrow[r] & E(2)^{16*}(RP^{2n} \times RP^{16K+9})
\end{tikzcd}$$

From Don Davis's work, the image of $(u_1 +_F u_2)^{2^{L-1}-n} \in ER(2)^{16*}(RP^\infty \times RP^\infty)$ in $E(2)^{16*}(RP^{2n} \times RP^{16K+10})$ is non-zero. We need to show that the image in $ER(2)^{16*}(RP^{2n} \times RP^{16K+9})$ is non-zero, (Note that we cannot use $E(2)$-cohomology for this purpose, as it is complex-oriented, which implies that $u^{8K+5} \in E(2)^*(RP^{16K+10})$ maps to zero in $E(2)^*(RP^{16K+9})$.

The two end terms in $(u_1 +_F u_2)^{2^{L-1}-n}$ are $u_1^{2^{L-1}-n}$ and $u_2^{2^{L-1}-n}$, which are plainly zero; all the other terms have the form $\lambda \alpha^k u_1^i u_2^j$, where $\lambda \in \mathbb{Z}_{(2)}, k \geq 0, i \geq 1, j \geq 1,$

and $i+j \geq 2^{L-1}-n$. Following [12], we may use the formulae

$$2u_1 = -\alpha u_1^2 + \dots, \ \ \alpha u_1 u_2^2 = \alpha u_1^2 u_2 + \dots$$

and induction to reduce $(u_1 +_F u_2)^{2^{L-1}-n}$ to a sum of distinct terms of the forms $\alpha^k u_1^i u_2$ and $u_1^i u_2^j$, with no numerical cofficient. (The first formula comes from $[2](u_1) = 0$; the second from $u_1[2](u_2) - u_2[2](u_1) = 0$.) Further, again by [12], $u_1^{n+1} = 0$ since $n \equiv 0$ or 7 mod 8. Then $\alpha^k u_1^i u_2 = 0$, since we still have $i+1 \geq 2^{L-1}-n$. We do not know (or need to know) exactly which terms are present; all we have to do is show that the monomials $u_1^i u_2^j$ (for $1 \leq i \leq n$ and $1 \leq j \leq 8K+5$) remain linearly independent in $ER(2)^*(RP^{2n} \times RP^{16K+9})$, which we defer to the next section.

Meanwhile, let us review the various numerical conditions. We need $n = m + \alpha(m) - 1$, $k = 2m - \alpha(m)$,$k \equiv 2$ mod 8 and $n \equiv 0$ or 7 mod 8. So $2m - \alpha(m) \equiv 2$ and $m + \alpha(m) \equiv 0$ or 1. Solving these, we get $(m, \alpha(m)) \equiv$ (6,2) or (1,0).

Theorem 7.1. *If* $(m, \alpha(m)) \equiv$ *(6,2) or (1,0) mod 8,*

$RP^{2(m+\alpha(m)-1)}$ *does not immerse in* $\mathbb{R}^{2(2m-\alpha(m))+1}$.

7.1 Products with an odd space

We shall look into the Bockstein spectral sequence for

$$ER(2)^*(RP^{2n} \wedge RP^{16K+9}, *)$$

where $2n < 16K+9$.

The E^1-term is the usual

$$E(2)^*(RP^{2n} \wedge RP^{16K+9}, *)$$

$$\simeq E(2)^*(RP^{2n},*) \otimes E(2)^*(RP^{16K+9},*) \oplus \Sigma^{-16(8K+4)-1} E(2)^*(RP^{2n},*)$$

(from [7]) Also, we know that

$$E(2)^*(RP^{16K+9},*) \cong E(2)^*(RP^{16K+8},*) \oplus E(2)^*(S^{16K+9},*).$$

Since $E(2)^*(S^{16K+9},*)$ is free it does not affect the Tor term, only the tensor product. So our E^1-term is:

$$E(2)^*(RP^{2n},*) \otimes E(2)^*(RP^{16K+8},*) \oplus E(2)^*(RP^{2n},*) \otimes E(2)^*(S^{16K+9},*)$$

$$\oplus \Sigma^{16K-17} E(2)^*(RP^{2n},*)$$

A 2-adic basis for this is given by

$$v_2^s \alpha^k u_1^i u_2 \quad (0 \leq k, \quad 0 < i \leq n, \quad s < 8)$$

$$v_2^s u_1^i u_2^j \quad (0 < i \leq n, \quad 1 < j \leq 8K+4, \quad s < 8)$$

by the same reduction as before, and

$$v_2^s \alpha^k i_{16K+9} \quad (0 \leq k, \quad 0 < i \leq n, \quad s < 8)$$

$$v_2^s \alpha^k u_1^i z_{16K-33} \quad (0 \leq k, \quad 0 \leq i < n, \quad s < 8).$$

We know that $x i_{16K+9}$ represents $v_2^4 u_2^{8K+4}$. So $v_2^4 x u_1^n i_{16K+9}$ represents $u_1^n u_2^{8K+5}$. There is no differential on $u_1^n i_{16K+9}$. Also there is no differential on $v_2^4 u_1^n i_{16K+9}$. All we have to do is show that $u_1^n i_{16K+9}$ is not in the image of d^1. Since d^1 has even degree we only have to worry about the odd degree elements since $u_1^n i_{16K+9}$ is odd degree.

d^1 has degree 18 so if it is to hit $u_1^n i_{16K+9}$ it must start at some $\alpha^k u_1^i z_{16K-33}$

because they are the only elements in the correct degree modulo 16. Then we would have d^1 non-trivial on z_{16K-33}.

In the Bockstein spectral sequence for $ER(2)^*(RP^{16M+16} \wedge RP^{16K+10}, *)$, $8M+8 < 8K+5$, we have from Theorem 19.2 [12], $d^1(z_{16K-1}) = 0$. From Theorem 1.2 of [7] we have that z_{16K-1} maps to $u_1 z_{16K-33}$ in the spectral sequence for $ER(2)^*(RP^{16K+16} \wedge RP^{16K+10}, *)$. Since this passes through the spectral sequence for $ER(2)^*(RP^{16M+16} \wedge RP^{16K+9}, *)$, z_{16K-1} maps to $u_1 z_{16K-33}$ here as well. So $d^1(u_1 z_{16K-33}) = u_1 d^1(z_{16K-33}) = 0$.

$$\begin{tikzcd}
ER(2)^*(RP^{16K+16} \wedge RP^{16K+10}) \arrow[r] \arrow[d] & ER(2)^*(RP^{16M+16} \wedge RP^{16K+8}) \\
ER(2)^*(RP^{16M+16} \wedge RP^{16K+9}) \arrow[ur] &
\end{tikzcd}$$

All elements killed by multiplication by u_1 go to zero under the map to $RP^{16K} \wedge RP^{16K+9}$, so our $d^1(z_{16K-33})$ is zero.

Theorem 7.2. *When* $n \leq 8M < 8M+8 < 8K+5$, *in*

$$ER(2)^{16*}(RP^{2n} \wedge RP^{16K+9})$$

the element $u_1^n u_2^{8K+5}$ *is non-zero.*

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