On Spaces of Matrices Satisfying Some Rank Conditions

by

Zoran Z. Petrović

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Abstract

In this work we investigate linear spaces of matrices, all of which have either constant rank or rank not less than some fixed number. These spaces are closely connected to nonsingular bilinear maps, which are used in proving immersions of real projective spaces. We use, as a main tool, Stiefel-Whitney characteristic classes and real K-theory to give some bounds for the dimension of these spaces. In some cases of matrices of constant low rank, we give the exact dimensions of these spaces.

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Introduction

It is a longstanding problem to determine, for a given n, the smallest possible k, such that real projective n-space $\mathbb{R}P^n$ can be immersed into $\mathbb{R}^{n+k}$ (see, e.g. [J], [G], [D]). One of the most powerful tools for proving immersability of projective spaces is the construction of nonsingular bilinear maps (see e.g. [Milg], [L]). Motivated by this problem, we investigate some linear spaces of matrices, containing, in addition to a zero matrix, only matrices of fixed rank or rank not less that some fixed number.

In the first section, we give some results on the bases of spaces of such matrices. This extends the results from [Han].

In the second section, we use the construction from [S] and [Mesh] to give a simple proof of a result from [ALP] on the maximum possible dimension of spaces of invertible matrices. We use real K-theory to prove this result. We also determine the maximum possible dimension of spaces of matrices of rank 2, using Stiefel-Whitney characteristic classes.

In the third section, we give some upper bounds for spaces of $n \times n$ matrices of rank at least $n-1$, using the ideas from [S].

In the last section, we show the relation between the problem of the existence of nonsingular bilinear maps and the maximum possible dimension of spaces of matrices of maximal rank, as well as the more general relations of these dimensions to geometric dimensions of vector bundles over real projective spaces. We also determine the maximum dimension of spaces of matrices of rank 3 and 7.

1 Bases of spaces of matrices

Let us denote by $M(m,n;\mathbb{R})$ the space of all $m \times n$ matrices over the field of real numbers $\mathbb{R}$. We shall investigate the following question:

Let $1 \leq k \leq n$. What is the maximal possible dimension of a vector subspace of $M(m,n;\mathbb{R})$ containing no non-zero matrices of rank less than k?

Remark 1 *Of course, if $k = 1$ we get the whole space.*

The importance of this question lies in the following.

1) Let $k = n = m$. We are then interested in finding the maximum possible dimension of a vector subspace of $M(n,n;\mathbb{R})$ consisting entirely of invertible matrices (except for the zero matrix). As pointed out in [ALP], this corresponds to the problem of vector fields on spheres. So, we know the answer: the maximum possible dimension is $\rho(n)$, where $\rho(n)$ is the Hurwitz-Radon number.

2) Let $k = 2$. In this case we want to find the maximum possible dimension of a vector subspace of $M(m,n;\mathbb{R})$ containing no matrices of rank 1. This corresponds to the problem of the existence of nonsingular bilinear maps.

Definition 2 *Let $f : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^k$ be a bilinear map. We say that f is nonsingular if the following condition holds:*

$$f(x,y) = 0 \Rightarrow x = 0 \text{ or } y = 0.$$

The question is: For a given m and n, what is the smallest possible k such that there exists a map as defined above?

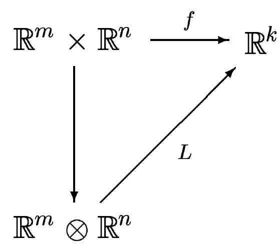
The solution to this problem has an immediate topological application due to the following (see, e.g. [J]):

Theorem 3 *If there exists a nonsingular bilinear map*

$$f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^k$$

then $\mathbb{R}P^{n-1}$ *immerses in* $\mathbb{R}^{k-1}$.

Now, suppose we have a bilinear map $f : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^k$. This map induces a linear map $L : \mathbb{R}^m \otimes \mathbb{R}^n \to \mathbb{R}^k$:



But $\mathbb{R}^m \otimes \mathbb{R}^n \cong M(m,n;\mathbb{R})$ and the map L has the property

$$L(x \otimes y) = 0 \Rightarrow x = 0 \text{ } or \text{ } y = 0$$

if and only if the map f is nonsingular.

Now, the elements in $\mathbb{R}^m \otimes \mathbb{R}^n$ of the form $x \otimes y$ correspond to matrices of rank≤ 1. So the nonsingularity of f is equivalent to the following property of L:

* Ker(L) does not contain any matrix of rank 1.

On the other hand, if V is any subspace of $M(m,n;\mathbb{R}) \cong \mathbb{R}^m \otimes \mathbb{R}^n$ which does not contain a matrix of rank one, the projection

$$\pi : M(m,n;\mathbb{R}) \to M(m,n;\mathbb{R})/V$$

will satisfy the property mentioned above, and so we will get a nonsingular bilinear map

$$f : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^{mn-\dim(V)}.$$

So, the maximum dimension of such a V corresponds to the minimum dimension in the problem of nonsingular bilinear maps.

We shall now extend the results of Handel [Han], namely we prove the following:

Theorem 4 *Let $2 \leq k \leq m, n$, and let q be the largest integer for which $M(m, n; \mathbb{R})$ admits a vector subspace of dimension q that does not contain non-zero elements of rank $< k$. Then, there exists such a V which admits a basis consisting only of matrices of rank k.*

This extends Theorem B in [Han] where it is proved only for the case $k = 2$.

We use the following definition, as in [Han]:

Definition 5 *Let V be a vector subspace of $M(m, n; \mathbb{R})$ and $e = \{A_1, \ldots, A_k\}$ be a basis of V. Define*

$$r(e) = \sum_i \operatorname{rank}(A_i)$$

Then e is called unshrinkable if

$$r(e) = \min\{r(e') : e' \textit{ is a basis of } V\}.$$

Let us also denote $r(V) = \min\{\operatorname{rank}(A) : A \in V, A \neq 0\}$. Then the following lemmas hold [Han]:

Lemma 6 *Let $e = \{A_1, \ldots, A_s\}$ be an unshrinkable basis of V. Then*

$$\operatorname{rank}\Big(\sum_i \alpha_i A_i\Big) \geq \max\{\operatorname{rank}(A_i) : \alpha_i \neq 0\}.$$

Lemma 7 *Let $e = \{A_1, \ldots, A_s\}$, $f = \{B_1, \ldots, B_s\}$ be bases of V such that $\operatorname{rank}(A_1) \leq \cdots \leq \operatorname{rank}(A_s)$ and $\operatorname{rank}(B_1) \leq \cdots \leq \operatorname{rank}(B_s)$. Assume e is unshrinkable. Then, $\operatorname{rank}(A_i) \leq \operatorname{rank}(B_i)$ for all i, with equality throughout if and only if f is unshrinkable.*

Lemma 8 *Suppose $r(V) = m$. Let $e = \{A_1, \ldots, A_t\}$ be a linearly independent subset of V with $\operatorname{rank}(A_i) = m$ for $1 \leq i \leq t$. Then e can be extended to an unshrinkable basis of M.*

Before proving our theorem, we first prove the following.

Proposition 9 *Let $k \geq 2$ be an integer and V be a subspace of $M(m, n; \mathbb{R})$ which is maximal among those which contain no non-zero elements of rank $< k$ ($k \geq 2$). Let $\{A_1, \ldots, A_s\}$ be an unshrinkable basis of V. Then*

$$\operatorname{rank}(A_i) \in \{k, k+1, \ldots, 3k-3\}.$$

Proof. Let V' be a subspace of V spanned by all the A_i's such that

$$\operatorname{rank}(A_i) \in \{k, k+1, \ldots, 3k-3\}.$$

Then, by Lemma 6, if $X \in V \backslash V'$, $\operatorname{rank}(X) > 3k - 3$.

Let $X \in M(m, n; \mathbb{R})$. Then, since V is maximal among these W such that $r(W) \geq k$, we have that the space spanned by V and X must contain an element of rank$< k$. So, there exist Z_1 and Y_1 such that

$$Z_1 = \alpha X + Y_1$$

where $\operatorname{rank}(Z_1) < k$, $Y_1 \in V$ and $\alpha \in \mathbb{R}$. Since $Z_1 \notin V$, $\alpha \neq 0$ and we can conclude that our X can be expressed in the form

$$X = Y + Z,\ Y \in V,\ \operatorname{rank}(Z) < k.$$

If $X = Y' + Z'$ is another expression for X with $\operatorname{rank}(Z') < k$ and $Y' \in V$, we have

$$Z - Z' = Y' - Y \in V$$

$$\operatorname{rank}(Z - Z') \leq \operatorname{rank}(Z) + \operatorname{rank}(Z') \leq k - 1 + k - 1 = 2(k-1)$$

and so $Z - Z' \in V'$. This shows that the following function is well defined:

$$f : M(m, n; \mathbb{R}) \to M(m, n; \mathbb{R})/V'$$

$$f(X) = Z + V' \quad \text{if} \;\; X = Y + Z,\; Y \in V,\;\; \operatorname{rank}(Z) < k.$$

Let us show that f is a linear map. Suppose

$$X = Y + Z, \quad Y \in V, \quad \operatorname{rank}(Z) < k.$$

Then

$$cX = cY + cZ, \qquad cY \in V, \qquad \operatorname{rank}(cZ) < k$$

$$f(X) = Z + V'$$

$$f(cX) = cZ + V'$$

and so $f(cX) = cf(X)$. Suppose

$$X = Y + Z, \quad Y \in V, \quad \operatorname{rank}(Z) < k$$

$$X' = Y' + Z', \quad Y' \in V, \quad \operatorname{rank}(Z') < k$$

$$X + X' = Y'' + Z'', \quad Y'' \in V, \quad \operatorname{rank}(Z'') < k.$$

Then

$$f(X) + f(X') = Z + Z' + V'$$

$$f(X + X') = Z'' + V'.$$

Now,

$$Z'' - Z - Z' = Y + Y' - Y'' \in V$$

and

$$\mathrm{rank}(Z'' - Z - Z') \leq 3(k-1).$$

So

$$Z'' - Z - Z' \in V'$$

and, consequently,

$$f(X + X') = f(X) + f(X').$$

So, we have a linear map

$$f : M(m, n; \mathbb{R}) \to M(m, n; \mathbb{R})/V'.$$

Moreover,

$$\begin{aligned}
\mathrm{Ker}(f) &= \{X = Y + Z : f(X) = 0\} \\
&= \{X = Y + Z : Z + V' = V'\} \\
&= \{X = Y + Z : Z \in V'\} \\
&= \{X \in V\} \\
&= V.
\end{aligned}$$

We show that f is onto: Let $Z \in M(m, n; \mathbb{R})$ be an arbitrary matrix of rank 1. Then

$$f(Z) = f(0 + Z) = Z + V'$$

Since every matrix is a sum of matrices of rank 1, we conclude that f is onto.

So, f induces an isomorphism:

$$\bar{f} : M(m, n; \mathbb{R})/V \cong M(m, n; \mathbb{R})/V'$$

and $\dim(V) = \dim(V')$, so $V = V'$ $(V' \subseteq V)$. □

Remark 10 *Of course, if $3k - 3 \geq n$, the previous proposition is trivial.*

Proof of Theorem 4. Let V be a subspace of $M(m,n;\mathbb{R})$ of maximum possible dimension which does not contain matrices of rank less than k. Let now $e = \{A_1, \ldots, A_q\}$ be an unshrinkable basis of V. If, for all i, $\mathrm{rank}(A_i) = k$ we are done. Otherwise, assume that $\mathrm{rank}(A_q) = l > k$. To every matrix A in $M(m,n;\mathbb{R})$ we can associate a linear operator $L_A : \mathbb{R}^n \to \mathbb{R}^m$ such that $\mathrm{rank}(A) = \mathrm{rank}(L_A)$. Now,

$$l = \mathrm{rank}(A_q) = \mathrm{rank}(L_{A_q}) = \dim(\mathrm{Im}(L_{A_q})).$$

We split $\mathrm{Im}(L_{A_q})$ as a direct sum:

$$\mathrm{Im}(L_{A_q}) = U \oplus W,$$

where $\dim(U) = k$ and $\dim(W) = l - k$. Let us denote by L' (resp. L''), the composition of L_{A_q} with the projection onto U (resp. W). Then it is clear that $\mathrm{rank}(L') = k$ and $\mathrm{rank}(L'') = l - k$. If we denote by A'_q (resp. A''_q) the matrix which corresponds to L' (resp. L''), we have that $\mathrm{rank}(A'_q) = k$, $\mathrm{rank}(A''_q) = l - k$ and $A'_q + A''_q = A_q$ (since $L_{A_q} = L' + L''$). Denote by $e' = \{A_1, \ldots, A_{q-1}, A'_q\}$. We want to prove that e' is a linearly independent subset such that the subspace spanned by e' does not contain matrices of rank less that k. Then, by Lemma 8, there exists an unshrinkable basis for this subspace which has more elements of rank k than our e. By repeating this process we eventually get a subspace of the same dimension with a basis consisting only of matrices of rank k, which is what we wanted to prove.

So suppose that some linear combination of elements in e' is of rank less than k (this will take care of both properties which we want to establish).

$$C = \sum_{i=1}^{q-1} \alpha_i A_i + \beta A'_q, \ \ \mathrm{rank}(C) \leq k - 1.$$

If $\beta = 0$ then $C = 0$, since there are no non-zero matrices of rank less than k in V. But, then all α_i's are zero. So $\beta \neq 0$ and we define a matrix D by:

$$D = \sum_{i=1}^{q-1} \alpha_i A_i + \beta A_q.$$

Since $\beta \neq 0$, $\operatorname{rank}(D) \geq l = \operatorname{rank}(A_q)$ by Lemma 6. On the other hand:

$$D = C + \beta(A_q - A'_q).$$

Then:

$$\begin{aligned} \operatorname{rank}(D) &\leq \operatorname{rank}(C) + \operatorname{rank}(A_q - A'_q) \\ &\leq k - 1 + l - k \\ &= l - 1. \end{aligned}$$

So, we have a contradiction and we are done. □

Remark 11 *This theorem holds true (with the same proof) for an arbitrary field, although the actual dimensions might be different.*

2 On spaces of matrices of fixed rank

Before proceeding further, let us introduce some notation. For $k \leq m, n$ we denote by $l(m, n; k)$ (resp. $L(m, n; k)$) the highest possible dimension of a vector subspace of $M(m, n; \mathbb{R})$ which consists entirely of matrices of rank$= k$ (resp. $\geq k$) (and a zero matrix of course). We give some estimates for these numbers.

Let us first deal with the case of matrices of constant rank. It is easy to see that $l(m, n; k) \geq \max\{m, n\} - k + 1$. Namely, the following matrix has rank k, for any choice of $x_1, \ldots, x_{n-k+1}$ (we assume that $m \leq n$) except when all of them are zero, in which case we have a zero matrix.

$$\begin{bmatrix} 0 & \cdots & 0 & x_1 & x_2 & \cdots & \cdots & x_{n-k+1} \\ 0 & \cdots & x_1 & x_2 & \cdots & \cdots & x_{n-k+1} & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ x_1 & x_2 & \cdots & \cdots & \cdots & x_{n-k+1} & \cdots & 0 \\ 0 & 0 & \cdots & \cdots & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \cdots & \cdots & 0 & \ldots & 0 \end{bmatrix}$$

All rows of this matrix except for the first k rows are zero, so the rank of this matrix is at most k. If $x_1 \neq 0$, then the determinant of the submatrix formed from the first k rows and columns of our matrix is non-zero and the rank is k. If $x_1 = 0$ we proceed to x_2 etc.

In order to give some upper bounds for $l(m, n; k)$ we use the following construction, which was introduced for the first time in [S] for the complex case and used in [W], [Mesh], [LY] for the complex and real case.

Suppose we have r matrices $A_1, \ldots, A_r$ in $M(m, n; \mathbb{R})$. We define a bundle map

$$n\xi_{r-1} \xrightarrow{f} \varepsilon^m$$

where by ξ_{r-1} we denote the canonical line bundle over $\mathbb{R}P^{r-1}$, and by ε^m, the trivial bundle over the same space, as follows:

$$f([x]\,; \lambda_1 x, \ldots, \lambda_n x) = ([x]\,; (x_1 A_1 + \cdots + x_r A_r) \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{bmatrix})$$

where $[x]$ is the class of $x = (x_1, \ldots, x_r) \in S^{r-1}$ in $\mathbb{R}P^{r-1}$.

If our matrices $A_1, \ldots, A_r$ all belong to a vector subspace of matrices of constant rank k and are linearly independent, then $\mathrm{Im}(f)$ becomes a vector bundle of dimension k, and we have the following splittings:

$$\mathrm{Im}(f) \oplus \nu^{m-k} = \varepsilon^m \tag{1}$$

$$\mathrm{Im}(f) \oplus \eta^{n-k} = n\xi_{r-1} \tag{2}$$

This idea was used in [Mesh] to prove that $l(n,n;k) \leq n$ (as a matter of fact, one can show similarly that $l(m,n;k) \leq \max\{m,n\}$), and also in [LY] to determine $l(n,n;n-1)$, $l(n,n;n-2)$ and $l(n,n-1;n-2)$. For $l(n,n;n)$ Lam and Yiu refer to the paper of Adams on vector fields on spheres [A]. In [ALP] the authors have carried out the determination of the maximum possible dimension of a real vector subspace of $M(n,n;\mathbb{R})$, $M(n,n;\mathbb{C})$, $M(n,n;\mathbb{H})$ of invertible matrices, referring again to [A] for the real case. As a matter of fact, one can give a proof of this result which does not use the whole machinery developed in [A], but only the fact on the structure of the KO group of $\mathbb{R}P^s$. This corresponds to 'linear' vector fields on spheres (those which come from linear operators).

Theorem 12 *$l(n,n;n) = \rho(n)$.*

Proof. Let us recall that $\rho(n)$ is the Hurwitz-Radon number defined by:

$$\rho(2^{4a+b}(2m+1)) = 8a + 2^b$$

where $n = 2^{4a+b}(2m+1)$, $a \geq 0$, $0 \leq b \leq 3$, $m \geq 0$ where a, b and m are integers. In [Hur] and [R] the construction of a space of invertible matrices of dimension $\rho(n)$ was performed, so we only need to prove that $l(n,n;n) \leq \rho(n)$. So, suppose that we have $l(n,n;n) \geq \rho(n)+1$. This means that there exist matrices $A_1, \ldots, A_{\rho(n)+1}$ such that all non-trivial linear combinations of them are invertible. They give us a bundle map

$$n\xi_{\rho(n)} \xrightarrow{f} \varepsilon^n$$

over $\mathbb{R}P^{\rho(n)}$. But, since the rank of f is everywhere n, $k = n$ in (1) and (2) and we have an isomorphism $n\xi_{\rho(n)} \cong \varepsilon^n$. This means that one has

$$n(\xi_{\rho(n)} - \varepsilon^1) = 0 \tag{3}$$

in $\widetilde{\mathrm{KO}}(\mathbb{R}P^{\rho(n)})$. We now recall the fact (see e.g. [A]) that

$$\widetilde{\mathrm{KO}}(\mathbb{R}P^s) \cong \mathbb{Z}/2^{\phi(s)}$$

generated by $\xi_s - \varepsilon^1$, where

$$\phi(s) \overset{\text{def}}{=} \operatorname{card}\{1 \leq m \leq s : m \equiv 0, 1, 2, 4 \pmod 8\}$$

Let us now write n in the form

$$n = 2^{4a+b}(2m+1),$$

where $a \geq 0$, $0 \leq b \leq 3$, $m \geq 0$ are integers. Then from (3)

$$(2m+1)2^{4a+b}(\xi_{\rho(n)} - \varepsilon^1) = 0 \tag{4}$$

in $\widetilde{\mathrm{KO}}(\mathbb{R}P^{\rho(n)})$, and since this is a 2-group we have that

$$2^{4a+b}(\xi_{\rho(n)} - \varepsilon^1) = 0 \tag{5}$$

in $\widetilde{\mathrm{KO}}(\mathbb{R}P^{\rho(n)})$. On the other hand,

$$\widetilde{\mathrm{KO}}(\mathbb{R}P^{\rho(n)}) = \mathbb{Z}/2^{\phi(\rho(n))}$$

generated by $\xi_{\rho(n)} - \varepsilon^1$, and

$$\phi(\rho(n)) = \phi(\rho(2^{4a+b}(2m+1))) = \phi(8a + 2^b).$$

Hence

$$\phi(\rho(n)) = \begin{cases} 4a+1, & b=0 \\ 4a+2, & b=1 \\ 4a+3, & b=2 \\ 4a+4, & b=3 \end{cases}$$

and $\phi(\rho(n)) > 4a + b$, which contradicts (5). So $l(n, n; n) \leq \rho(n)$ and we are done. □

Let us now proceed to investigate things 'on the other end', namely we prove the following theorem.

Theorem 13 *Let* $2 \leq m \leq n$, *and* $n \neq 3$. *Then*

$$l(m, n; 2) = n, \text{ if } n = 2k$$

$$l(m, n; 2) = 2k, \text{ if } n = 2k + 1.$$

Proof. It is easy to see that $l(m, 2k; 2) = 2k$. Namely, let us look at the vector space consisting of matrices of the form:

$$\begin{bmatrix} x_1 & -x_2 & x_3 & -x_4 & \cdots & x_{2k-1} & -x_{2k} \\ x_2 & x_1 & x_4 & x_3 & \cdots & x_{2k} & x_{2k-1} \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

where $x_i \in \mathbb{R}$. We see that every such matrix has rank=2, except when all of the x_i's are zero and we get the zero matrix. So, $l(m, 2k; 2) \geq 2k$ and, since we know that $l(m, 2k; 2) \leq 2k$ (as mentioned before), we are done. We show, similarly, that $l(m, 2k + 1; 2) \geq 2k$, by adding a column of 0's.

Let us now prove that $l(m, 2k+1; 2) = 2k+1$ is impossible. Otherwise we get the bundle map

$$(2k+1)\xi_{2k} \xrightarrow{f} \varepsilon^m$$

and splittings

$$\mathrm{Im}(f) \oplus \nu^{m-2} = \varepsilon^m \tag{6}$$

$$\mathrm{Im}(f) \oplus \eta^{2k-1} = (2k+1)\xi_{2k} \tag{7}$$

Now,

$$w_{2k}((2k+1)\xi_{2k}) = \binom{2k+1}{2k} u^{2k} = u^{2k} \neq 0$$

in $H^{2k}(\mathbb{R}P^{2k}; \mathbb{Z}/2)$, so that $(2k+1)\xi_{2k}$ cannot have two linearly independent sections. So $\mathrm{Im}(f)$ is not isomorphic to ε^2. Now we use the result from [Lev]: Every 2-bundle over $\mathbb{R}P^s$ for $s > 2$ splits into a sum of line bundles. Since every line bundle over $\mathbb{R}P^{2k}$ is isomorphic to either ξ_{2k} or ε^1, we now have two cases:

1) $\mathrm{Im}(f) = \xi_{2k} \oplus \xi_{2k}$. If by $w(\zeta)$ we denote the total Stiefel-Whitney class of a vector bundle ζ, we have, from (6):

$$w(\mathrm{Im}(f))w(\nu^{m-2}) = w(\varepsilon^m)$$
$$w(\xi_{2k} \oplus \xi_{2k})w(\nu^{m-2}) = 1$$
$$(1+u^2)w(\nu^{m-2}) = 1$$
$$w(\nu^{m-2}) = 1 + u^2 + \cdots + u^{2k}$$

So we have that $w_{2k}(\nu^{m-2}) \neq 0$, but $m-2 \leq 2k-1$ and this class must vanish.

2) $\mathrm{Im}(f) = \xi_{2k} \oplus \varepsilon^1$. From (7):

$$w(\mathrm{Im}(f))w(\eta^{2k-1}) = w((2k+1)\xi_{2k})$$

$$w(\xi_{2k} \oplus \varepsilon^1)w(\eta^{2k-1}) = (1+u)^{2k+1}$$

$$(1+u)w(\eta^{2k-1}) = (1+u)^{2k+1}$$

$$w(\eta^{2k-1}) = (1+u)^{2k}.$$

So $w_{2k}(\eta^{2k-1}) = u^{2k} \neq 0$ in $H^{2k}(\mathbb{R}P^{2k}; \mathbb{Z}/2)$ which is impossible and we are done. □

Remark 14 *$l(2,3;2) = 2$ and $l(3,3;2) = 3$ and so the assertion of the previous theorem is false for $n = 3$.*

Proof. If $l(2,3;2) = 3$ we would get that $\mathrm{Im}(f) = \varepsilon^2$ and $\mathrm{Im}(f) \oplus \eta^1 = 3\xi_2$. Then $w(\mathrm{Im}(f)) = 1$ and

$$w(\eta^1) = w(3\xi_2) = 1 + u + u^2$$

which is impossible.

The space of matrices of the form:

$$\begin{bmatrix} 0 & x_1 & x_2 \\ -x_1 & 0 & x_3 \\ -x_2 & -x_3 & 0 \end{bmatrix}$$

shows that $l(3,3;2) \geq 3$. □

3 On spaces of matrices of large rank

We now proceed to give some upper bounds for the numbers $L(m,n;k)$. We denote by $\nu_2(n)$ the highest power of 2 which divides n:

$$\nu_2(n) = \nu_2(2^s(2m+1)) = s.$$

Theorem 15 *If $n = 4k+1$, $L(n,n;n-1) \leq 2^{\nu_2(n-1)} + 1$.*

We first prove the following lemma.

Lemma 16 *Let* n *be odd and* $f : n\xi_{r-1} \to \varepsilon^n$ *be given as before, using matrices* $A_1, \ldots, A_r$*, which form a basis of a subspace of matrices of rank at least* $n-1$*. Define*

$$X_{n-1} \stackrel{def}{=} \{[x] \in \mathbb{R}P^{r-1} : \det(x_1 A_1 + \cdots + x_r A_r) = 0\}.$$

Then, if we denote by i *the inclusion:*

$$i : X_{n-1} \hookrightarrow \mathbb{R}P^{r-1},$$

$i^*(u)^{r-2} \neq 0$ *in* $H^*(X_{n-1}; \mathbb{Z}/2)$.

Proof. Our f induces a map:

$$n\xi_{r-1} \mid_{X_{n-1}} \stackrel{f'}{\longrightarrow} \varepsilon^n \mid_{X_{n-1}}$$

where f' has rank $n-1$ everywhere. Since $\text{codim}(\mathbb{R}P^1) + \text{codim}(X_{n-1}) = n$, we can compute the intersection product

$$\left[\mathbb{R}P^1\right] \cdot [X_{n-1}] \in \mathbb{Z}/2,$$

where by $[Y]$ we denote the fundamental class of Y. (Every algebraic variety has a fundamental class, even if it has singularities.) Since X_{n-1} is an algebraic variety, given as the zero set of a homogeneous polynomial of odd degree n, and this polynomial has an odd number of zeroes on a generic projective line $\mathbb{R}P^1$, we have:

$$\left[\mathbb{R}P^1\right] \cdot [X_{n-1}] \neq 0.$$

But

$$\left[\mathbb{R}P^1\right] \cdot [X_{n-1}] = \langle i^*(u^{r-2}), [X_{n-1}] \rangle,$$

since u^{r-2} is the Poincaré dual of $\left[\mathbb{R}P^1\right]$ in $\mathbb{R}P^{r-1}$ and we therefore conclude that $i^*(u)^{r-2} \neq 0$. $\square$

Proof of the theorem. Assume that $L(n,n;n-1) > 2^{\nu_2(n-1)}+1$. Note that $\nu_2(n-1) = \nu_2(4k) \geq 2$. This means that we can choose matrices $A_1, \ldots, A_r$, $r = 2^{\nu_2(n-1)}+2$ to define our bundle map. Then, as before

$$\mathrm{Im}(f') \oplus \nu^1 = \varepsilon^n \mid_{X_{n-1}} \tag{8}$$

$$\mathrm{Im}(f') \oplus \eta^1 = n\xi_{r-1} \mid_{X_{n-1}} . \tag{9}$$

So

$$w(\mathrm{Im}(f'))w(\nu^1) = 1 \tag{10}$$

$$w(\mathrm{Im}(f'))w(\eta^1) = (w(\xi_{r-1} \mid_{X_{n-1}}))^n \tag{11}$$

We get that

$$w(\eta^1) = (1+i^*(u))^n w(\nu^1) \tag{12}$$

The terms in degree 2 give:

$$0 = \binom{n}{2} i^*(u)^2 + \binom{n}{1} i^*(u) w_1(\nu^1) \tag{13}$$

Since $n = 4k+1$, $\binom{n}{2} = 0$ and $\binom{n}{1} = 1$ in $\mathbb{Z}/2$, and we have

$$i^*(u)w_1(\nu^1) = 0.$$

So $w_1(\nu^1) = 0$, since $i^*(u) \neq 0$ and, consequently (12) simplifies to:

$$w(\eta^1) = (1+i^*(u))^n.$$

Since $i^*(u)^{r-2} \neq 0$ by Lemma 16, we conclude that

$$\binom{n}{2} \equiv \binom{n}{3} \equiv \cdots \equiv \binom{n}{r-2} \equiv 0 \pmod 2.$$

Now, $n-1 = 2^s(2m+1)$ for some integers s and m, so that $r-2 = 2^s$. Let us denote $i^*(u)$ by x. We then have:

$$\begin{aligned}(1+x)^n &= (1+x)^{2^s(2m+1)+1} \\ &= (1+x^{2^s})^{2m+1}(1+x) \\ &= (1+x^{2^s}+\cdots)(1+x) \\ &= 1+x+x^{2^s}+\cdots\end{aligned}$$

where we used $\cdots$ to denote the higher order terms. From this we see that

$$\binom{n}{r-2} \equiv 1 \pmod 2$$

and we are done. □

Let us now look at the subspace V of maximum possible dimension which consists only of $m \times n$ matrices of rank $\geq k \geq 2$ and the zero matrix. So $\dim(V) = L(m,n;k)$. Let us define a linear map

$$L : V \longrightarrow M(m, n-1; \mathbb{R})$$

by stripping the last column as follows:

$$L(A) = L([B \mid C]) = B$$

if $A = [B \mid C]$, where $A \in V$, $B \in M(m, n-1; \mathbb{R})$ and $C \in M(m, 1; \mathbb{R})$.

Claim 17 $\mathrm{Ker}(L) = \{0\}$.

Proof. Otherwise, let $0 \neq A \in \mathrm{Ker}(L)$. This means that A is of the form $A = [0 \mid C]$, where $C \neq 0$. But, this would mean that $\mathrm{rank}(A) = 1$ which is impossible. □

So $\dim(V) = \dim(\mathrm{Im}(L))$, but $\mathrm{Im}(L)$ contains, except for the zero matrix, only matrices of rank at least $k-1$:

$$\mathrm{rank}(A) = \mathrm{rank}([B \mid 0] + [0 \mid C]) \leq \mathrm{rank}(B) + 1.$$

So $\mathrm{rank}(B) \geq \mathrm{rank}(A) - 1 \geq k-1$. From this we conclude that

$$L(m,n;k) \leq L(m,n-1;k-1)$$

as long as $k \geq 2$. Similarly,

$$L(m,n;k) \leq L(m-1,n;k-1).$$

By applying repeatedly these two inequalities we can conclude that

$$L(m,n;k) \leq L(m-s,n-t;k-s-t) \leq (m-s)(n-t)$$

for all s,t such that $s+t \leq k-1$. If we assume that $m \leq n$, the best upper bound is achieved for $s = k-1$ and $t = 0$. In that case we have

$$L(m,n;k) \leq (m-k+1)n.$$

In the paper [Rees], the author has, using some arguments from algebraic geometry, given the exact value

$$L(n,n;n-1) = 4 \quad \text{for } n = 4k+2$$

which, by the way, is the answer we get for the complex case and any n (see [S]). Such a simple answer cannot be given in other cases, e.g. for $n = 4k+1$ we have:

$$L(4k+1,4k+1;4k) \geq l(4k+1,4k+1;4k) = \rho(4k).$$

The last equality follows from the results in [LY]. So we see that we can make $L(n,n;n-1)$ arbitrarily large if n is of the form $4k+1$. The same thing is true for two other cases and the case in [Rees] is by far the easiest one.

4 Nonsingular bilinear maps

In this section we proceed to explain relations between the numbers $l(m, n; k)$ and the existence of nonsingular bilinear maps, as well as their connection to the geometric dimension of vector bundles over real projective spaces. We also give some new bounds for these numbers.

As pointed out in [LY], the determination of the numbers $l(m, n; n)$ for $m \geq n$ corresponds to the question of the existence of nonsingular bilinear maps. More precisely:

Theorem 18 *There exists a nonsingular bilinear map*

$$f : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^{n+k}$$

if and only if $l(n + k, n; n) \geq n$.

Instead of proving this theorem we prove a theorem which is slightly more general.

Theorem 19 *There exists a nonsingular bilinear map*

$$f : \mathbb{R}^m \times \mathbb{R}^n \longrightarrow \mathbb{R}^k$$

if and only if $l(k, n; n) \geq m$ *or* $l(k, m; m) \geq n$.

Proof. Suppose there exists a nonsingular bilinear map

$$f : \mathbb{R}^m \times \mathbb{R}^n \longrightarrow \mathbb{R}^k.$$

Then this map induces a map

$$L : \mathbb{R}^m \longrightarrow \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^k)$$

as follows:

$$L(x)(y) = f(x, y).$$

This map has two properties:

1) L is mono: If $L(x) = 0$ then for any $y \neq 0$ in $\mathbb{R}^n$

$$f(x, y) = L(x)(y) = 0$$

and since f is nonsingular, $x = 0$.

2) If $x \neq 0$ then $L(x)$ is mono: Suppose that $L(x)(y) = 0$. Then

$$f(x, y) = L(x)(y) = 0$$

and since f is nonsingular and $x \neq 0$ we have $y = 0$.

Property 2) shows that $\mathrm{Im}(L)$ contains only operators of rank n, while property 1) shows that

$$m = \dim(\mathbb{R}^m) = \dim(\mathrm{Im}(L))$$

and we conclude that $l(k, n; n) \geq m$.

One proves the converse in a similar way. $\square$

Definition 20 *We say that a vector bundle η^l has geometric dimension $\leq k$:*

$$\text{g.d.}(\eta^l) \leq k$$

if it is stably equivalent to a k-bundle, i.e. there exist m and n such that

$$\eta^l \oplus \varepsilon^m \cong \zeta^k \oplus \varepsilon^n$$

for some k-bundle ζ^k.

Remark 21 *Note that when two bundles are stably equivalent, they have the same total Stiefel-Whitney class.*

Since every bundle over $\mathbb{R}P^s$ is stably equivalent to $m\xi_s$ for some m (this follows from the structure of the group $\widetilde{\mathrm{KO}}(\mathbb{R}P^s)$), in order to determine the geometric dimension of vector bundles over $\mathbb{R}P^s$, we only need to consider bundles of the form $m\xi_s$. The following theorem gives the connection between spaces of matrices of constant rank and the geometric dimension of these bundles.

Theorem 22 *If $m \geq n$ and $l(m,n;n) \geq r$ then*

$$\text{g.d.}(m\xi_{r-1}) \leq m - n.$$

Proof. As before we construct a map

$$m\xi_{r-1} \xrightarrow{f} \varepsilon^n$$

and get a splitting

$$m\xi_{r-1} \cong \varepsilon^n \oplus \eta^{m-n},$$

since $\mathrm{Im}(f) = \varepsilon^n$. □

We know that $l(n+k,n;n) \geq n$ is equivalent to the existence of a nonsingular bilinear map

$$f : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^{n+k}.$$

Since $l(n+k,n;n) = l(n,n+k;n)$ (this corresponds to taking the transpose of a matrix), the fact that $l(n+k,n;n) \geq n$ produces two maps:

$$(n+k)\xi_{n-1} \xrightarrow{f_1} \varepsilon^n \tag{14}$$

$$n\xi_{n-1} \xrightarrow{f_2} \varepsilon^{n+k} \tag{15}$$

and splittings:

$$(n+k)\xi_{n-1} \cong \varepsilon^n \oplus \eta^k \tag{16}$$

$$\varepsilon^{n+k} \cong n\xi_{n-1} \oplus \nu^k. \tag{17}$$

These splittings correspond to each other by tensoring with ξ_{n-1}, so we see that this operation is related to the operation of transposing a matrix.

Note that, since

$$n\xi_{n-1} \cong \tau(\mathbb{R}P^{n-1}) \oplus \varepsilon^1$$

(see, e.g. [MS]), the tangent bundle $\tau(\mathbb{R}P^{n-1})$ has a stable normal k-bundle, and so by a theorem of Hirsch (see [Hir]), $\mathbb{R}P^{n-1}$ can be immersed into $\mathbb{R}^{n+k-1}$. So, the construction we have used many times can also provide a proof of the statement mentioned in the first section on the relation between nonsingular bilinear maps and immersions of projective spaces.

Let us now give some bounds for numbers $l(m, n; k)$.

Theorem 23 *Suppose that $k \leq \rho(n)$. Then*

$$l(mn + r, s; k) \geq mn + \max\{r - k + 1, 0\}$$

for $0 \leq r < n$ and $k \leq s \leq mn + r$.

This theorem is a simple consequence of the following lemma and the earlier result $l(m, n; k) \geq \max\{m, n\} - k + 1$.

Lemma 24 *Let $k \leq \rho(n)$. Then*

$$l(n, k; k) = n.$$

Proof. There exists a nonsingular bilinear map

$$f : \mathbb{R}^{\rho(n)} \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$$

since $l(n, n; n) = \rho(n)$, and by restriction we get a nonsingular bilinear map

$$f' : \mathbb{R}^k \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$$

which shows that $l(n,k;k) \geq n$. □

Proof of the Theorem. Let us first observe that

$$l(s+t,k;k) \geq l(s,k;k) + l(t,k;k)$$

since this corresponds to the formation of a block matrix in $M(s+t,k;\mathbb{R})$ from the matrices in $M(s,k;\mathbb{R})$ and $M(t,k;\mathbb{R})$. We now have:

$$\begin{aligned} l(mn+r,s;k) &\geq l(mn+r,k;k) \\ &\geq ml(n,k;k) + l(r,k;k) \\ &= mn + l(r,k;k). \end{aligned}$$

Since $l(r,k;k) = 0$ if $r < k$ and $l(r,k;k) \geq r-k+1$ if $r \geq k$, we are done. □

In order to give some upper bounds for these numbers we first prove the following lemma.

Lemma 25 *For $n \geq 1$ and any $(2^n - 1)$-bundle η over $\mathbb{R}P^{s-1}$ we have that*

$$w(\eta) = w(r\xi_{s-1})$$

for some r, $0 \leq r \leq 2^n - 1$.

Proof. As noted previously,

$$w(\eta) = w(k\xi_{s-1})$$

for some k. Let us write k in the form:

$$k = 2^n q + r, \quad 0 \leq r < 2^n.$$

Then

$$w(k\xi_{s-1}) = (1+u)^k = (1+u^{2^n})^q (1+u)^r.$$

Since $w_i(\eta) = 0$ for $i > 2^n - 1$ we conclude that

$$w(\eta) = (1+u)^r = w(r\xi_{s-1})$$

and we are done. □

Theorem 26 *$l(2^k m + r, 2^k m + r; 2^k - 1) \leq 2^k m$ for $0 \leq r \leq 2^k - 2$.*

Proof. Suppose that

$$l(2^k m + r, 2^k m + r; 2^k - 1) \geq 2^k m + 1.$$

Then, as before, we can construct a map

$$(2^k m + r)\xi_{2^k m} \xrightarrow{f} \varepsilon^{2^k m + r}$$

and get splittings:

$$\mathrm{Im}(f) \oplus \nu \cong \varepsilon^{2^k m + r} \quad (18)$$

$$\mathrm{Im}(f) \oplus \eta \cong (2^k m + r)\xi_{2^k m} \quad (19)$$

where ν and η are $(2^k(m-1)+r+1)$-bundles. Since $\mathrm{Im}(f)$ is a (2^k-1)-bundle, by the previous lemma we have:

$$w(\mathrm{Im}(f)) = (1+u)^s \quad \text{for } 0 \leq s \leq 2^k - 1.$$

Case 1) $s = 2^k - 1$. From the first splitting:

$$\begin{aligned} (1+u)^{2^k-1} w(\nu) &= 1 \\ (1+u)^{2^k} w(\nu) &= 1+u \\ w(\nu) &= (1+u)(1+u^{2^k} + \cdots + u^{2^k m}) \end{aligned}$$

and so $w_{2^k m}(\nu) \neq 0$, which is impossible since

$$2^k(m-1) + r + 1 \leq 2^k m - 1.$$

Case 2) $0 \leq s < 2^k - 1$. The second splitting then gives:

$$\begin{aligned} (1+u)^s w(\eta) &= (1+u)^{2^k m + r} \\ w(\eta) &= (1+u)^{2^k m + r - s}. \end{aligned}$$

Since

$$2^k m + r - s \geq 2^k m + r - 2^k + 2 > 2^k m + r - 2^k + 1$$

we again get a contradiction. □

Combining these results we can get a complete description of the numbers $l(n, n; 3)$ and $l(n, n; 7)$.

Theorem 27 *Let $k \geq 0$. Then:*

$$\begin{aligned} l(4k, 4k; 3) = l(4k+1, 4k+1; 3) = l(4k+2, 4k+2; 3) &= 4k \\ l(4k+3, 4k+3; 3) &= 4k+1 \\ l(8k, 8k; 7) = l(8k+1, 8k+1; 7) = \cdots = l(8k+6, 8k+6; 7) &= 8k \\ l(8k+7, 8k+7; 7) &= 8k+1. \end{aligned}$$

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Vita

Zoran Z. Petrović was born in Belgrade, Yugoslavia on October 16, 1965. He received his Bachelor's degree from the School of Natural Sciences and Mathematics, Belgrade University in 1988. Since then he has been appointed as an Assistant at this School. He received his degree of Master of Science from the same School upon writing a thesis 'Generalized cohomological index theory' in 1991. He started his doctoral studies at The Johns Hopkins University in 1991 when he was awarded an Owen Fellowship. He received his M.A. degree from The Johns Hopkins University in 1992. During the academic year 1994/95 he was on leave of absence which he spent teaching at Belgrade University.