Dieudonné Theory for Ungraded and Periodically Graded Hopf Rings

Rui Miguel Saramago

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Abstract

Hopf rings are algebraic objects that appear regularly in homotopy theory. They can for example arise as homology modules of certain spaces, for certain generalized homology theories. It becomes thus important to have available efficient methods for computing Hopf ring structures in different environments.

One way to devise such methods is to construct functors from a category of coalgebras, for which Hopf rings are the ring objects, to other categories where it might become easier to perform calculations and deal with the structure. Some of these categories are categories of Dieudonné modules. Hopf rings correspond thus to ring objects in these categories, and such objects are called Dieudonné rings.

This work proposes a study of ungraded and periodically graded Dieudonné rings. These correspond, in the associated category of coalgebras, to ungraded Hopf rings, and generalize the known constructions for the graded case. This generalization is important because, even though certain useful homology theories – like singular homology and various bordisms – do give rise to graded Hopf rings and thus to graded Dieudonné rings, there are nonetheless important generalized homology theories that do not carry a natural grading.

Morava K-theories, on the other hand, can be viewed as a middle term in this discussion: they do carry some grading, but it is periodic and thus of a different nature from the grading that appears, for example, in singular homology.

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Chapter 1

Background material on category theory

This chapter settles some basic notions on category theory that will be used later. We define abelian group objects and abelian ring objects for some categories. In this work we will typically be looking at such kinds of objects for some special categories of coalgebras.

Suppose $\mathcal{C}$ is a category with finite products Π. Given two objects X and Y in $\mathcal{C}$, write $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ for the set of morphisms from X to Y. The element $1_X \in \mathrm{Hom}_{\mathcal{C}}(X, X)$ will denote the identity morphism for the object X in $\mathcal{C}$.

In this work, we further assume that $\mathcal{C}$ has a *terminal object*, i. e. a N in $\mathcal{C}$ such that, for each X in $\mathcal{C}$, the set $\mathrm{Hom}_{\mathcal{C}}(X, N)$ contains exactly one morphism ε_X (It may be regarded as the product of the empty family).

An *abelian group object* of $\mathcal{C}$ is an $X \in \mathcal{C}$ together with maps $\eta \in \mathrm{Hom}_{\mathcal{C}}(N, X)$ (*abelian group unit*, or *zero*), $* \in \mathrm{Hom}_{\mathcal{C}}(X \Pi X, X)$ (*addition*) and $\chi \in \mathrm{Hom}_{\mathcal{C}}(X, X)$ (*inverse*). These have to satisfy the following commutative diagrams:

$$\begin{tikzcd}
N\Pi X \arrow[r, "p_2"] \arrow[d, "\eta\Pi 1_X"'] & X \arrow[d, "1_X"] \\
X\Pi X \arrow[r, "*"] & X
\end{tikzcd}$$

(addition by zero)

$$\begin{tikzcd}
X\Pi X \arrow[dd, "(p_2, p_1)"'] \arrow[dr, "*"] & \\
& X \\
X\Pi X \arrow[ur, "*"] &
\end{tikzcd}$$

(commutativity)

$$\begin{tikzcd}
X\Pi X\Pi X \arrow[r, "1_X\Pi *"] \arrow[d, "*\Pi 1_X"'] & X\Pi X \arrow[d, "*"] \\
X\Pi X \arrow[r, "*"] & X
\end{tikzcd}$$

(associativity)

$$\begin{tikzcd}
X \arrow[r, "(1_X, \chi)"] \arrow[d, "\varepsilon_X"'] & X\Pi X \arrow[d, "*"] \\
N \arrow[r, "\eta"] & X
\end{tikzcd}$$

(existence of inverses)

We will deal in this work with abelian group objects for several categories of coalgebras, and these will be called Hopf algebras.

Consider for instance the category $\mathcal{CA}_k$ of cocommutative coassociative coalgebras with counit over a ring k with unit. For each object C in $\mathcal{CA}_k$ we have a coproduct $\Psi_C\colon C \to C \otimes_k C$ and a counit $\varepsilon_C\colon C \to k$. The category has k as a terminal object, with natural coproduct and counit. The category also has finite products: $C\Pi D$ is defined as $C \otimes_k D$ and, given maps $f\colon B \to C$ and

$g\colon B \to D$, the map $(f,g)\colon B \to C \otimes_k B$ is given by $(f \otimes g) \circ \Psi_B$.

The abelian group objects for this category $\mathcal{CA}_k$ form the category of *commutative Hopf algebras over k.* Such an Hopf algebra is thus a cocommutative coassociative coalgebra with counit over k, together with an "addition" (or *product*) $* \in \mathrm{Hom}_{\mathcal{CA}_k}(X \otimes_k X, X)$ and a "zero" $\eta \in \mathrm{Hom}_{\mathcal{CA}_k}(k, X)$ that satisfy the previous diagrams. In particular, the coproduct and product have to be compatible in a precise way:

$$\begin{tikzcd}
X \otimes X \arrow[r, "\Psi\otimes\Psi"] \arrow[d, "*"'] & X \otimes X \otimes X \otimes X \arrow[r, "1\otimes T\otimes 1"] & X \otimes X \otimes X \otimes X \arrow[d, "*\otimes*"] \\
X \arrow[rr, "\psi"] & & X \otimes X
\end{tikzcd}$$

(Here $T : X \otimes X \to X \otimes X$ is the switching map taking $x \otimes y \in X \otimes X$ to $y \otimes x$)

$$\begin{tikzcd}
k \arrow[r, "\psi"] \arrow[d, "\eta"'] & k \cong k \otimes k \arrow[d, "\eta\otimes\eta"] \\
X \arrow[r, "\psi"] & X \otimes X
\end{tikzcd}$$

Given still a category $\mathcal{C}$ with finite products, we can also consider its *commutative graded ring objects with unit.* These are by definition a collection $X_* = \{X_n\}_{n\in\mathbb{Z}}$ of abelian group objects of $\mathcal{C}$ together with maps $\circ = \circ_{ij} \in \mathrm{Hom}_{\mathcal{C}}(X_i \Pi X_j, X_{i+j})$ (multiplication) and $e \in \mathrm{Hom}_{\mathcal{C}}(N, X_0)$ (multiplicative unit) such that:

$$\Psi_X = (1_X, 1_X) \in \mathcal{C}(X, X\Pi X)$$

$$\begin{tikzcd}
X_i \Pi X_j \Pi X_k \arrow[r, "1_{X_i} \Pi \circ"] \arrow[d, "\circ \Pi 1_{X_k}"'] & X_i \Pi X_{j+k} \arrow[d, "\circ"] \\
X_{i+j} \Pi X_k \arrow[r, "\circ"] & X_{i+j+k}
\end{tikzcd}$$

(associativity)

$$\begin{tikzcd}
X_i \Pi X_j \arrow[r, "\circ"] \arrow[d, "(p_2, p_1)"'] & X_{i+j} \arrow[d, "\chi^{ij}"] \\
X_j \Pi X_i \arrow[r, "\circ"] & X_{i+j}
\end{tikzcd}$$

(commutativity)

$$\begin{tikzcd}
X_i \Pi X_j \Pi X_j \arrow[r, "\psi \Pi 1_{X_j \Pi X_j}"] \arrow[d, "(1_{X_i}) \Pi *_j"'] & X_i \Pi X_i \Pi X_j \Pi X_j \arrow[r, "(p_1, p_3, p_2, p_4)"] & X_i \Pi X_j \Pi X_i \Pi X_j \arrow[d, "\circ \Pi \circ"] \\
X_i \Pi X_j \arrow[r, "\circ"] & X_{i+j} & X_{i+j} \Pi X_{i+j} \arrow[l, "*_{i+j}"']
\end{tikzcd}$$

(distributivity)

$$\begin{tikzcd}
N \Pi X_i \arrow[r, "p_2"] \arrow[d, "e \Pi 1_{X_i}"'] & X_i \arrow[d, "1_{X_i}"] \\
X_o \Pi X_i \arrow[r, "\circ"] & X_i
\end{tikzcd}$$

(multiplication by the unit)

$$\begin{tikzcd}
N \Pi X_j \arrow[r, "\varepsilon"] \arrow[d, "\eta \Pi 1_{X_j}"'] & N \arrow[d, "\eta"] \\
X_i \Pi X_j \arrow[r, "\circ"] & X_{i+j}
\end{tikzcd}$$

(multiplication by zero)

As before, we concentrate on several categories of coalgebras and study their graded ring objects, which we will call Hopf rings.

For the category $\mathcal{CA}_k$, for instance, we get as Hopf rings collections of Hopf algebras $\{X_k\}_{k\in\mathbb{Z}}$ together with a multiplication

$$\circ = \circ_{ij} \in \mathrm{Hom}_{\mathcal{CA}_k}(X_i \otimes_k X_j, X_{i+j})$$

and a multiplicative unit $e \in \mathrm{Hom}_{\mathcal{CA}_k}(k, X_0)$ that satisfy the corresponding commutative diagrams.

Further information on Hopf rings can be found in [12].

For the basic result in Dieudonné theory, we will need two more category definitions.

An *additive category* $\mathcal{C}$ is one that contains a zero object 0 (i. e., an object that is both an initial and a terminal object of $\mathcal{C}$) and, for each two objects A and B of $\mathcal{C}$, a binary composition $+$ defined on $\mathrm{Hom}_{\mathcal{C}}(A, B)$ making $(\mathrm{Hom}_{\mathcal{C}}(A, B), +, 0_{AB})$ an abelian group. (Here 0_{AB} is defined as $0_{0B}0_{A0}$, where 0_{0B} is the unique map $0_{0B} : 0 \to B$ and 0_{A0} is the unique map $0_{A0} : A \to 0$; these maps are given from the definition of both initial and terminal object.) We further impose that each binary composition be distributive with respect to the original product (composition of maps) on $\mathcal{C}$. Also, $\mathcal{C}$ must have finite products.

An *abelian category* $\mathcal{C}$ is an additive category such that each morphism in $\mathcal{C}$ has a kernel and a cokernel, every monic is a kernel of its cokernel and every epic is a cokernel of its kernel (in this general context, a *kernel* for a morphism $f : A \to B$ in $\mathcal{C}$ is a morphism $k : S \to A$ satisfying $fk = 0_{SB}$ and every

$h : C \to A$ with $fh = 0_{CB}$ factors uniquely through k. The notion of *cokernel* is defined similarly; it is dual to the notion of kernel).

Chapter 2

Background material on Hopf algebras

In this section we discuss some special Hopf algebras that will be used in the construction of Dieudonné modules for given Hopf algebras.

Fix a prime p and consider the p-adic integers $\mathbb{Z}_p$. Let $x_0, x_1, \cdots$ be a sequence of indeterminates. We focus on some particular elements in $\mathbb{Z}_p[x_0, x_1, \cdots]$, the free commutative algebra over $\mathbb{Z}_p$ in the indeterminates x_i:

Definition 2.1. The *Witt polynomials* ω_n, for $n \geq 0$, are given by

$$\omega_n(x) = x_0^{p^n} + px_1^{p^{n-1}} + \cdots + p^n x_n$$

where $x = (x_0, x_1, \cdots)$

The Witt polynomials are important for the next result.

Theorem 2.2. *[6] There exists a unique Hopf algebra structure on the algebra $\mathbb{Z}_p[x_0, x_1, ...]$ such that the Witt polynomials ω_n are primitive.*

From now on, whenever we refer to the *Hopf algebra* $\mathbb{Z}_p[x_0, x_1, \cdots]$ we mean the free commutative algebra over the indeterminates together with the unique coproduct that makes the Witt polynomials primitive.

We can also consider just the algebra $\mathbb{Z}_p[x_0, x_1, \cdots, x_k]$. In this case, the coproduct defined from Theorem 2.2 restricts to a coproduct in this finitely generated algebra, and we will call $CW(k)$ the Hopf algebra $\mathbb{Z}_p[x_0, x_1, \cdots, x_k]$ together with the restricted coproduct.

If we want to work in the graded case, start by giving each x_i degree $p^i m$ for some fixed m and then define $CW_m(k)$ to be the graded Hopf algebra that corresponds to $CW(k)$.

Proposition 2.3. *[6] Let* $[p] : \mathbb{Z}_p[x_0, x_1, \cdots] \to \mathbb{Z}_p[x_0, x_1, \cdots]$ *be* p*-times the identity map in the abelian group of Hopf algebra maps* $\mathbb{Z}_p[x_0, x_1, \cdots] \to \mathbb{Z}_p[x_0, x_1, \cdots]$. *Then* $[p](x_i) \equiv x_{i-1}^p (\mathrm{mod} p)$

Next we want to consider Hopf algebras over a perfect field $\mathbb{F}_p$ with characteristic p. Define Hopf algebras $H(k) = \mathbb{F}_p \otimes CW(k) = \mathbb{F}_p[x_0, x_1, \cdots, x_k]$. In the graded case, write $H(n) = \mathbb{F}_p[x_0, x_1, \cdots, x_k]$, where $n = p^k m$ for $(p, m) = 1$ and each x_i has degree $p^i m$. We will make no other distinctions in the notation for the graded and ungraded case. It will be clear from context if we mean that $H(k)$ has or does not have an associated grading.

Definition 2.4. For a Hopf algebra H over $\mathbb{F}_p$, the Frobenius is the homomorphism $F : H \to H$ taking an element x to the element x^p. The Verschiebung $V : H \to H$ is the dual to the Frobenius in the dual algebra.

The Verschiebung can be described as follows: If an element $x \in H$ has p-fold coproduct $\psi^p(x) = \Sigma x' \otimes x' \otimes \cdots \otimes x' + \Sigma_{(\text{not all y equal})} y' \otimes y'' \otimes \cdots \otimes y^{(p+1)}$, then the Verschiebung on x is $V(x) = \Sigma x'$.

Since we are dealing with Hopf algebras over a perfect field $\mathbb{F}_p$, both the Verschiebung and the Frobenius are homomorphisms of Hopf algebras.

Chapter 3

Bilinear maps and tensor products

In this section we present general constructions that will be used in the proof of the equivalence between ring categories in various cases.

Let $\mathcal{C}$ be a category with finite products and $\mathcal{A} \subseteq \mathcal{C}$ a subcategory of abelian objects.

Since any $A \in \mathcal{A}$ is an abelian object, $F_A = \mathrm{Hom}_{\mathcal{C}}(\cdot, A)$ is a functor to abelian groups - see Chapter 1 (Note: The functor is defined on the opposite category $\mathcal{C}^{OP}$.)

Definition 3.1. If A, B and C are objects in $\mathcal{A}$, a morphism $\varphi : A \times B \to C$ in $\mathcal{C}$ is a *bilinear map* if for all $X \in \mathcal{C}$ the induced map

$$F_A(X) \times F_B(X) \to F_C(X)$$

is a natural bilinear map of abelian groups.

We are interested in *initial* bilinear maps.

Definition 3.2. If A and B are objects in $\mathcal{A}$, an initial bilinear map

$\varepsilon : A \times B \to A \boxtimes B$ in $\mathcal{A}$ is called a *tensor product* of A and B. That is, if $\varphi : A \times B \to C$ is any bilinear map, there is a unique morphism $\psi : A \boxtimes B \to C$ in $\mathcal{A}$ making the diagram

$$\begin{tikzcd}
A \times B \arrow[rr, "\varepsilon"] \arrow[dr, "\varphi"'] & & A \boxtimes B \arrow[dl, "\psi"] \\
& C &
\end{tikzcd}$$

commute in $\mathcal{C}$.

As with any initial objects of a category, if $A \boxtimes B$ exists it is unique up to isomorphism in $\mathcal{A}$.

Theorem 3.3. *[8] Assume that the categories $\mathcal{A}$ and $\mathcal{C}$ satisfy*

a) both $\mathcal{C}$ and $\mathcal{A}$ have all limits and colimits;

b) the forgetful functor $\mathcal{A} \to \mathcal{C}$ has a left adjoint S.

Then any two objects $A, B \in \mathcal{A}$ have a tensor product $A \boxtimes B$ in $\mathcal{A}$.

If $\mathcal{C}$ is a category of coalgebras over a commutative ring k and $\mathcal{A} \subseteq \mathcal{C}$ is a category of bicommutative Hopf algebras, a bilinear map becomes a morphism of coalgebras $\varphi : H_1 \otimes_k H_2 \to K$ between *Hopf algebras* H_1, H_2 and K. This morphism has to reduce to a bilinear map of abelian groups once we apply the functor $\mathrm{Hom}_{\mathcal{C}}(\cdot, X)$ for any *coalgebra* X. In these cases, we have tensor products $H_1 \otimes H_2 \to H_1 \boxtimes H_2$ [6].

Chapter 4

Dieudonné theory for graded Hopf algebras

This chapter will deal with some categories of graded Hopf algebras over $\mathbb{F}_p$, their Dieudonné modules and Dieudonné rings. All our Hopf algebras will be bicommutative, that is, in each case the product will be commutative and the coproduct will be co-commutative. We call such a Hopf algebra *connected* if $H_0 \cong \mathbb{F}_p$. It will be called *group-like in degree zero* if $H_0 \cong \mathbb{F}_p[G]$, a group ring. Connected Hopf algebras are group-like in degree zero.

4.1 Graded connected Hopf algebras and their Dieudonné modules

In this section we will consider the category $\mathcal{HA}_*$ of graded, connected, bicommutative Hopf algebras over $\mathbb{F}_p$. The main results can be found in [14], [11] and [6].

Consider the Hopf algebras $H(n) = \mathbb{F}_p[x_0, \cdots, x_k]$, where $n = p^k m$ for $(p, m) = 1$ and each x_i has degree $p^i m$. These Hopf algebras form a set of projective generators for $\mathcal{HA}_*$ [5].

We have a morphism $v : H(n) = \mathbb{F}_p[x_0, \cdots, x_k] \to \mathbb{F}_p[x_0, \cdots, x_{k+1}] = H(pn)$ given by inclusion. Also, by Proposition 2.3, there exists a unique map of Hopf algebras $f : H(pn) \to H(n)$ making the following diagram commute.

$$\begin{tikzcd}
H(pn) \arrow[r, "f"] \arrow[dr, "{[p]}"'] & H(n) \arrow[d, "v"] \\
 & H(pn)
\end{tikzcd}$$

This map satisfies $vf = [p]$ and also $fv = [p]$.

We now define Dieudonné modules for Hopf algebras in $\mathcal{HA}_*$.

Definition 4.1. The *Dieudonné module* for a Hopf algebra $H \in \mathcal{HA}_*$ is the graded abelian group

$$\{D_n H\}_{n \geq 1} = \{\mathrm{Hom}_{\mathcal{HA}_*}(H(n), H)\}_{n \geq 1}$$

together with homomorphisms

$$F : D_n H \to D_{pn} H$$

and

$$V : D_{pn} H \to D_n H$$

The homomorphisms above come from the previous maps f and v by composition on the left:

$$\begin{tikzcd}
H(pn) \arrow[r, "f"] \arrow[rr, bend right, "F(\varphi)"'] & H(n) \arrow[r, "\varphi"] & H
\end{tikzcd}$$

$$\begin{tikzcd}
H(n) \arrow[r, "v"] \arrow[rr, bend right, "V(\varphi)"'] & H(pn) \arrow[r, "\varphi"] & H
\end{tikzcd}$$

These homomorphisms reflect thus, in Dieudonné modules, the Verschiebung and the Frobenius defined on Hopf algebras.

We have $VF = FV = p$ (Here, p stands for p times the identity map).

Also, by Proposition 2.3, if $n = p^s k$ with $(p, k) = 1$ then the order of the identity map in $\mathrm{Hom}_{\mathcal{HA}_*}(H(n), H(n))$ is p^{s+1}, and so $p^{s+1} D_n H = 0$.

We define a category of Dieudonné modules.

Definition 4.2. We call $\mathcal{DM}_*$ the category of graded modules M together with maps V, dividing degree by p, and F, multiplying degree by p, such that $FV = VF = p$.

Note: For V, it is assumed in the definition that, if the degree of $x \in M_i$ is not a multiple of p for some i, then $Vx = 0$. In particular, given any $x \in M$, $V^n x = 0$ for some $n \geq 0$.

The above category can also be defined as the category of graded modules M over $R = \mathbb{F}_p[F, V]/(FV - p)$, where we put $\deg(F) = 1$ and $\deg(V) = -1$ and define $\deg(ax) = p^{\deg(a)} \deg(x)$ for $a \in R$ and $x \in M$ (or zero, if this calculation provides a fraction).

We have defined thus a functor $D_* : \mathcal{HA}_* \to \mathcal{DM}_*$. This functor will provide an equivalence of categories.

The following theorem was proved by Schoeller and independently by Ravenel.

Theorem 4.3. *[14] [11] The above functor* D_* *has a right adjoint* $U_* : \mathcal{DM}_* \to \mathcal{HA}_*$*, and the pair* (D_*, U_*) *forms an equivalence of categories.*

The proof confirms in this case the fact that an abelian category with a set of small projective generators is equivalent to a category of modules over some ring. [3] [4]

4.2 Graded connected Hopf rings and their Dieudonné rings

In this section we consider ring objects in the category $\mathcal{HA}_*$ of graded connected Hopf algebras over $\mathbb{F}_p$, and look for corresponding ring objects in the category of Dieudonné modules.

As in the previous section, let $CW_m(k) = \mathbb{Z}_p[x_0, \cdots, x_k]$, where $n = p^k m$ with $(p, m) = 1$ and $H(n) = \mathbb{F}_p \otimes CW_m(k) \cong \mathbb{F}_p[x_0, \cdots, x_k]$. Define $G(n) = QCW_m(k)$, the indecomposables of $CW_m(k)$. We can view $G(n) \otimes G(n')$ as a $\mathbb{Z}_p[V]$ module for each n and n'. We have the following result.

Proposition 4.4. *[6] There exists an isomorphism*

$$R \otimes_{\mathbb{Z}_p[V]} (G(n) \otimes G(n')) \to D_*(H(n) \boxtimes H(n'))$$

where $R = \mathbb{F}_p[F, V]/(FV - p)$

This isomorphism takes the element $1 \otimes (x_k \otimes x_{k'}) \in R \otimes_{\mathbb{Z}_p[V]} (G(n) \otimes G(n'))$ into the element in $D_{k+k'}(H(n) \boxtimes H(n')) = \mathrm{Hom}_{\mathcal{HA}_*}(H(n+n'), H(n) \boxtimes H(n'))$ that gives to $x_{k+k'} \in H(n + n')$ the value $\varphi(x_k \otimes x_{k'})$, where $\varphi : H(n) \otimes H(n') \to H(n) \boxtimes H(n')$ is the tensor product map, and such that it commutes with Verschiebungs (details are in [6]). Call $\iota_n \otimes \iota_{n'}$ this element in $D_*(H(n) \boxtimes H(n'))$. These elements, for various values of n and n', will allow us to construct ring structures in Dieudonné modules.

Let now $H = \{H_n\}_{n\in\mathbb{Z}}$ be a Hopf ring in the case we are considering in this section. That is, $\{H_n\}_{n\in\mathbb{Z}}$ will be a graded ring object in the category $\mathcal{CA}_*$ of graded connected coassociative coalgebras over $\mathbb{F}_p$.

By Section 4.1, we know we can determine the corresponding Dieudonné module for each Hopf algebra H_n , and then get an algebraic object $\{D_*H_n\}_{n\in\mathbb{Z}}$. We are interested in determining how the ring structure in H carries over to $\{D_*H_n\}_{n\in\mathbb{Z}}$ and also how this new ring structure relates to the various homomorphisms V and F in the Dieudonné modules.

Consider then the bilinear pairing

$$\circ_{ij} : H_i \otimes H_j \to H_{i+j}$$

that is given by the ring structure of H. We can get an induced pairing

$$D_mH_i \times D_nH_j \to D_{m+n}H_{i+j}$$

as follows.

Given $x \in D_mH_i = \mathrm{Hom}_{\mathcal{HA}_*}(H(m), H_i)$ and $y \in D_nH_j = \mathrm{Hom}_{\mathcal{HA}_*}(H(n), H_j)$, consider the obvious map $x \otimes y : H(m) \otimes H(n) \to H_m \otimes H_n$. The composition

$$\begin{tikzcd}H(n) \otimes H(m) \arrow[r, "x\otimes y"] & H_i \otimes H_j \arrow[r] & H_{i+j}\end{tikzcd}$$

is a bilinear map, so by definition of tensor product $\boxtimes$ we can get a unique map of Hopf algebras $g : H(n) \boxtimes H(m) \to H_{i+j}$ that makes the following diagram commute as coalgebras:

$$\begin{tikzcd}
H(m) \otimes H(n) \arrow[r, "\varphi"] \arrow[d, "x\otimes y"'] & H(m) \boxtimes H(n) \arrow[d, "g"] \\
H_i \otimes H_j \arrow[r, "\circ_{ij}"] & H_{i+j}
\end{tikzcd}$$

(Here the top map is the one given in the definition of tensor product $\boxtimes$).

Note that this homomorphism depends on the original $x \in D_m H_i$ and $y \in D_n H_j$.

Using $g : H(m) \boxtimes H(n) \to H_{i+j}$ we can produce an element in $D_{m+n} H_{i+j}$. Apply the functor D_* to g and get

$$D_* g : D_*(H(m) \boxtimes H(n)) \to D_* H_{i+j}$$

$D_* g$ is given simply by composition with g on the left. That is, if $\alpha \in D_r(H(m) \boxtimes H(n)) = \mathrm{Hom}_{\mathcal{HA}_*}(H(r), H(m) \boxtimes H(n))$, then $D_* g(\alpha)$ is the map $H(r) \xrightarrow{\alpha} H(m) \boxtimes H(n) \xrightarrow{g} H_{i+j}$.

Using Proposition 4.4, we can obtain an element $D_* g(\iota_m \otimes \iota_n)$ in $D_{m+n} H_{i+j}$. It is given by the composition $H(m+n) \xrightarrow{\eta_m \otimes \eta_n} H(m) \boxtimes H(n) \xrightarrow{g} H_{i+j}$ taking $x_{k+k'}$ to $g(\varphi(x_k \otimes x_{k'}))$, where as before $\varphi : H(n) \otimes H(n') \to H(n) \boxtimes H(n')$ is the tensor product map.

We obtained pairings

$$D_m H_i \times D_n H_j \to D_{m+n} H_{i+j}$$

and these pairings finally produce a pairing

$$\circ' : D_* H_i \otimes D_* H_j \to D_* H_{i+j}$$

of Dieudonné modules.

We now want to determine how the homomorphisms V and F defined on each D_*H_i relate to the new ring structure put on them.

We have the following result.

Proposition 4.5. *[6] Given bilinear pairings* $\circ_{ij} : H_i \otimes H_j \to H_{i+j}$*, the induced bilinear pairings* $\circ'_{ij} : D_*H_i \otimes D_*H_j \to D_*H_{i+j}$ *satisfy:*

(a) $V(x \circ' y) = Vx \circ' Vy$;

(b) $Fx \circ' y = F(x \circ' Vy)$;

(c) $x \circ' Fy = F(Vx \circ' y)$.

With this information, it becomes natural to define a category of Dieudonné rings.

Definition 4.6. A *graded connected Dieudonné ring* over a perfect field $\mathbb{F}_p$ is a sequence $\{M_{*i}\}_{i\in\mathbb{Z}}$ of graded connected Dieudonné modules together with bilinear maps $\circ'_{ij} : M_{*i} \otimes M_{*j} \to M_{*i+j}$ satisfying the conditions in the previous proposition.

We are interested in obtaining universal bilinear pairings for Dieudonné modules in $\mathcal{DM}_*$.

Definition 4.7. A bilinear pairing in $\mathcal{DM}_*$ is a map $f : M_1 \times M_2 \to N$ of graded abelian groups satisfying $\deg(f(x,y)) = \deg(x) + \deg(y)$, $Vf(x,y) = f(Vx, Vy)$, $Ff(x, Vy) = f(Fx, y)$ and $Ff(Vx, y) = f(x, Fy)$.

Proposition 4.5 above stated that a bilinear pairing $\circ : H_1 \otimes H_2 \to K$ in $\mathcal{HA}_*$ induces a bilinear pairing $\circ' : D_*H_1 \otimes D_*H_2 \to D_*K$ in $\mathcal{DM}_*$. This is the reason for defining bilinear pairings in $\mathcal{DM}_*$ this way.

Proposition 4.8. *There exist universal bilinear pairings*

$$\eta : M_1 \times M_2 \to M_1 \boxtimes_{\mathcal{DM}_*} M_2$$

in $\mathcal{DM}_*$.

Proof. Consider the graded tensor product $M \otimes N$. This can be viewed as a $\mathbb{Z}_p[V]$ module by putting $V(x \otimes y) = Vx \otimes Vy$.

Consider the module $\mathbb{F}_p[F, V] \otimes_{\mathbb{Z}_p[V]} (M \otimes N)$ and the submodule K generated by the relations $F \otimes x \otimes Vy = 1 \otimes Fx \otimes y$ and $F \otimes Vx \otimes y = 1 \otimes x \otimes Fy$ for all $x \in M$ and $y \in N$. Now define $M \boxtimes_{\mathcal{DM}_*} N = \mathbb{F}_p[F, V] \otimes_{\mathbb{Z}_p[V]} (M \otimes N)/K$.

We claim that the map $\eta : M_1 \times M_2 \to M_1 \boxtimes_{\mathcal{DM}_*} M_2$ taking (x, y) to $1 \otimes x \otimes y$ is an initial bilinear pairing in $\mathcal{DM}_*$.

First we show it is bilinear. Since the degree condition is trivially verified, we consider the other relations. We have:

$$Vf(x, y) = V(1 \otimes x \otimes y) = 1 \otimes V(x \otimes y) = 1 \otimes Vx \otimes Vy = f(Vx, Vy)$$

$$Ff(x, Vy) = F(1 \otimes x \otimes Vy) = F \otimes x \otimes Vy = 1 \otimes Fx \otimes y = f(Fx, y)$$

$$Ff(Vx, y) = F(1 \otimes Vx \otimes y) = F \otimes Vx \otimes y = 1 \otimes x \otimes Fy = f(x, Fy)$$

Now we show it is initial. Given any bilinear pairing $\xi : M_1 \times M_2 \to N$, we define a map $\varphi : M_1 \boxtimes_{\mathcal{DM}_*} M_2 \to N$ by $\varphi(a \otimes x \otimes y) = a\xi(x, y)$ for $a \in \mathbb{F}_p[V, F]$, $x \in M_1$ and $y \in M_2$. This is a morphism of Dieudonné modules, since $\varphi(V(a \otimes x \otimes y)) = \varphi(Va \otimes V(x \otimes y)) = \varphi(Va \otimes Vx \otimes Vy) = Va\xi(Vx, Vy)$ □

Given an initial bilinear pairing $H \times K \to H \boxtimes K$ in $\mathcal{HA}_*$, we get an induced pairing in Dieudonné modules $D_*H \times D_*K \to D_*(H \boxtimes K)$ (see [6]).

This is a bilinear pairing, and thus we get an induced map $D_*H \boxtimes_{\mathcal{DM}_*} D_*K \to D_*(H \boxtimes K)$.

Theorem 4.9. *[6] For any Hopf algebras H and K in $\mathcal{HA}_*$, the induced map $D_*H \boxtimes_{\mathcal{DM}_*} D_*K \to D_*(H \boxtimes K)$ in $\mathcal{D}_*$ is an isomorphism.*

The proof of this theorem is long, technical, and involves analyzing special categories of torsion-free Hopf algebras over $\mathbb{Z}_p$ (those with a lift of the Verschiebung). We present the result here to show one way of proving the equivalence between ring categories of Hopf algebras and Dieudonné modules in the special case of Section 4.3. We will give an alternative, direct proof of this equivalence that does not rely on such techniques.

4.3 Dieudonné theory for graded Hopf algebras, group-like in degree zero

The universal bilinear map $\boxtimes$ on the previous category $\mathcal{HA}_*$ does not supply that category with a symmetric monoidal structure, as we don't have a unit object for $\boxtimes$.

We extend the category in order to include such an object and then we can prove the equivalence between the categories of Hopf rings and Dieudonné rings.

Suppose H is a non-negatively graded Hopf algebra over a commutative ring k and let $H_0 \subseteq H$ be the sub Hopf algebra of elements in degree zero. Let $H_c = k \otimes_{H_0} H$ be the connected component of the identity. We have

$H \cong H_c \otimes_k H_0$.

Definition 4.10. Call $\mathcal{HA}_*^o$ the category of bicommutative Hopf algebras over $\mathbb{F}_p$ that are group-like in degree zero; that is, of those for which H_0 in the above decomposition is a group ring.

Now define $H(0) = \mathbb{F}_p[\mathbb{Z}]$ and remember that, for $n > 0$, $H(n)$ was defined as $H(n) = \mathbb{F}_p \otimes CW_m(k) = \mathbb{F}_p[x_0, \cdots, x_k]$ for $n = p^k m$ with $(m,p) = 1$.

For a Hopf algebra H in $\mathcal{HA}_*^o$, define $D_*H = \{D_nH = \mathrm{Hom}_{\mathcal{HA}_*^o}(H(n), H)\}_{n \geq 0}$. We have homomorphisms $V : D_{pn}H \to D_nH$ and $F : D_nH \to D_{pn}H$ for $n > 0$ (given by the previous definitions) and, for $n = 0$, we know that $V : D_0H \to D_0H$ is the identity (since the Verschiebung is the identity on a group-ring) and also that $F : D_0H \to D_0H$ is p times the identity.

We can thus define a new category.

Definition 4.11. The category $\mathcal{DM}_*^o$ has as objects non-negatively graded abelian groups together with maps $V : M_{pn} \to M_n$ and $F : M_n \to M_{pn}$ satisfying

(a) $M_c = \{M_n\}_{n>0}$ is a Dieudonné module in $\mathcal{DM}_*$;

(b) $VF = FV = p$;

(c) V is the identity on M_0.

Theorem 4.12. *[6] The functor $D_* : \mathcal{HA}_*^o \to \mathcal{DM}_*^o$ gives an equivalence of categories.*

Let $\mathcal{CA}_*^o$ be the category of cocommutative coalgebras over $\mathbb{F}_p$ that are set-like in degree zero, i.e. of those that can be written as $C \cong C_c \otimes_k C_0$, with C_0

a ring. The category $\mathcal{HA}_*^o$ consists of the abelian group objects of $\mathcal{CA}_*^o$ and the forgetful functor $\mathcal{HA}_*^o \to \mathcal{CA}_*^o$ has a left adjoint [6], so by Theorem 3.3 there exist universal bilinear pairings $H \otimes K \to H \boxtimes K$ in $\mathcal{HA}_*^o$.

Proposition 4.13. *For every* $H \in \mathcal{HA}_*^o$ *we have an isomorphism* $\mathbb{F}_p[\mathbb{Z}] \boxtimes H \cong H$.

We also have universal bilinear pairings in $\mathcal{DM}_*^o$:

Similarly to what happened in Section 4.2, a bilinear pairing in $\mathcal{DM}_*^o$ is defined as a bilinear map of non-negatively graded abelian groups $f : M \times N \to K$ such that $f(Vx, Vy) = Vf(x, y)$, $f(Fx, y) = Ff(x, Vy)$ and $f(x, Fy) = Ff(Vx, y)$.

Then we have a map $M \times N \to M \boxtimes_{\mathcal{DM}_*^o} N$,
where $M \boxtimes_{\mathcal{DM}_*^o} N \simeq R \otimes_{\mathbb{Z}_p[V]} M \otimes N/K$, with K the submodule generated by $F \otimes (x \otimes Vy) - 1 \otimes (Fx \otimes y)$ and $F \otimes (Vx \otimes y) - 1 \otimes (x \otimes Fy)$ for all x and y.

A bilinear pairing $H_1 \otimes H_2 \to K$ in $\mathcal{HA}_*^o$ induces a pairing $\mu : D_*H_1 \times D_*H_2 \to D_*K$ in $\mathcal{HA}_*^o$ as follows:

If $x \in D_nH_1$ and $y \in D_mH_2$ with both n and m greater than zero, then Section 4.2 gives an element $\mu(x, y)$ in $D_{m+n}K$.

If either m or n is zero, the isomorphism $H \cong \mathbb{F}_p[\mathbb{Z}] \boxtimes H$ provides a map $H(n) \to H(n) \boxtimes H(0)$, and the diagram on page 20 also defines an element $\mu(x, y) \in D_nK$.

Theorem 4.14. *The universal pairing in* $\mathcal{HA}_*^o$*,* $H \otimes K \to H \boxtimes K$*, induces a pairing* $D_*H \times D_*K \to D_*(H \boxtimes K)$ *in* $\mathcal{DM}_*^o$*.*

This pairing is bilinear in $\mathcal{DM}_*^o$ *and induces an isomorphism* $D_*H \boxtimes_{\mathcal{DM}_*^o} D_*K \to D_*(H \boxtimes K)$.

Corollary 4.15. *The categories* $\mathcal{HA}_*^o$ *and* $\mathcal{DM}_*^o$ *are symmetric monoidal categories and the equivalence* $D_* : \mathcal{HA}_*^o \to \mathcal{DM}_*^o$ *is an equivalence of categories with symmetric monoidal structure.*

The previous Corollary allows us to prove that the ring objects of $\mathcal{HA}_*^o$ are equivalent to the ring objects of $\mathcal{D}_*^o$.

First, notice that in $\mathcal{HA}_*^o$ products are defined as

$$\tilde{\varphi} : (A_o \otimes A_c) \otimes (B_o \otimes B_c) \to (C_o \otimes C_c)$$

with $\tilde{\varphi}(ax \otimes by) = \varphi(a \otimes b)\varphi'(x \otimes y)$, where $\varphi : A_o \otimes B_o \to C_o$ is a product of group rings and $\varphi' : A_c \otimes B_c \to C_c$ is a pairing of graded connected Hopf algebras. (Here we write ax for $a \otimes x \in A_o \otimes A_c$, etc.)

In order to analyze bilinear maps in $\mathcal{DM}_*^o$, first notice that any $M \in \mathcal{DM}_*^o$ can be written as $M_o \otimes_{\mathbb{F}_p} M$, where $M \in \mathcal{DM}_*$ and M_o is the graded module concentrated in degree zero and for which V is the identity. A bilinear map in $\mathcal{DM}_*^o$ is then a map $\tilde{\varphi} : (M_o \otimes M) \otimes (N_o \otimes N) \to (K_o \otimes K)$ given by $\tilde{\varphi}(am \otimes bm) = \varphi(a \otimes b)\varphi'(m \otimes n)$, where $\varphi : M_o \otimes N_o \to K_o$ is a product in group rings and $\varphi' : M \otimes N \to K$ is a bilinear map in $\mathcal{DM}_*$ as defined before.

Following the reasoning from Section 4.2, any pairing $\circ_{ij} : H_i \otimes H_j \to H_{i+j}$ in $\mathcal{HA}_*^o$ will induce a pairing $\circ'_{ij} : D_*H_i \otimes D_*H_j \to D_*H_{i+j}$ in $\mathcal{DM}_*^o$. We simply notice that the diagram

$$\begin{tikzcd}
H(0) \otimes H(0) \arrow[r] \arrow[d, "a\otimes b"'] & H(0) \boxtimes H(0) \arrow[d, "g"] \\
(H_i)_0 \otimes (H_j)_0 \arrow[r, "\circ"] & (H_{i+j})_0
\end{tikzcd}$$

allows us to define products $\circ : (D_i)_0 \otimes (D_j)_0 \to (D_{i+j})_0$. We also have the induced products on the Dieudonné modules for the connected parts of the Hopf algebras H, given by the restriction to $\mathcal{HA}_*$ of the pairing $\circ$. Thus we define

$$ax \circ' by = (a \circ b)(x \circ y)$$

This definition takes care of the cases of $a \circ' y$ and $x \circ' b$, for we simply have $a \circ' y = a1 \circ' 1y$ and $x \circ' b = 1x \circ' b1$.

Let $\mathcal{HR}^o_*$ denote the category that has as objects sequences $\{H_i\}_{i\in\mathbb{Z}}$ of graded Hopf algebras that are group-like in degree zero together with pairings $H_i \otimes H_j \to H_{i+j}$ and $\mathcal{DR}^o_*$ the category whose objects are sequences $\{M_i\}_{i\in\mathbb{Z}}$ of Dieudonné modules in $\mathcal{D}^o_*$ together with pairings $M_i \otimes M_j \to M_{i+j}$ satisfying the conditions we had before.

Consider now the functor $D^{\mathcal{R}}_* : \mathcal{HR}^o_* \to \mathcal{DR}^o_*$ that takes each sequence of Hopf algebras $\{H_i\}_{i\in\mathbb{Z}}$ to the sequence of Dieudonné modules $\{D_*(H_i)\}_{i\in\mathbb{Z}}$, where D_* is the previous functor $D_* : \mathcal{HA}^o_* \to \mathcal{DM}^o_*$, and such that the products $\circ_{ij} : H_i \otimes H_j \to H_{i+j}$ carry over to the products $\circ'_{ij} : D_*H_i \otimes D_*H_j \to D_*H_{i+j}$ as given in the preceding paragraph.

Corollary 4.16. *The functor* $D^{\mathcal{R}}_* : \mathcal{HR}^o_* \to \mathcal{DR}^o_*$ *has a right adjoint* $U^{\mathcal{R}}_* : \mathcal{DR}^o_* \to \mathcal{HR}^o_*$ *and the pair* $(D^{\mathcal{R}}_*, U^{\mathcal{R}}_*)$ *forms an equivalence of categories.*

The proof follows from the fact that $\mathcal{HA}_*^o$ and $\mathcal{D}_*^o$ are equivalent categories, that both have a symmetric monoidal structure (given respectively by $\boxtimes$ and $\boxtimes_{\mathcal{D}_*}$) and that by the previous theorem the symmetric monoidal structure is preserved by the equivalence $D_* : \mathcal{HA}_*^o \to \mathcal{D}_*^o$.

4.4 An alternative proof of the equivalence between ring categories

This section offers an alternative proof of the equivalence between the two ring categories of the previous section, which dealt with the case of group-like in degree zero graded Hopf algebras. The method presented here involves the construction of an inverse functor for the previously given functor and the proof that the two functors do define an equivalence of categories.

Similar arguments can be used to prove the equivalence between ring categories in the graded connected case, which the previous proof did not allow us to do. The method of constructing an inverse functor will also be used in the next chapters, when we deal with ungraded and periodically graded Hopf algebras.

In the following results we will extend somehow the definition of Dieudonné module for a graded connected Hopf algebra: they will not be positively graded modules as before, but non-negatively graded ones, since we will define $D_0H = \mathrm{Hom}_{\mathcal{HA}_*}(H(0), H)$. (Here, as before, $H(0) = \mathbb{F}_p[\mathbb{Z}]$). Thus, for each graded connected Hopf algebra H in $\mathcal{HA}_*$, $D_0H \cong \mathbb{F}_p$ as a $\mathbb{F}_p$-module. This definition does not bring anything new and this category of non-negatively graded Dieudonné modules is easily seen to be equivalent to the previous one with positively graded ones, but with this new convention it will be easier to write down the following results.

We start with a lemma that offers a result symmetric to the one on Proposition 4.5.

Lemma 4.17. *Any bilinear pairing* $\circ_{ij} : D_*H_i \otimes D_*H_j \to D_*H_{i+j}$ *induces a bilinear pairing* $\bar{\circ}_{ij} : H_i \otimes H_j \to H_{i+j}$.

Proof. We focus on the primitive elements of $H_i \otimes H_j$.

Suppose $x \in H_i$ is a primitive element in degree m and consider the element $1 \in H_j$ in degree zero. Define the homomorphism $\hat{x} \in D_m H_i = \mathrm{Hom}_{\mathcal{HA}_*}(H(m), H_i)$ by $\hat{x}(1) = 1$, $\hat{x}(x_m) = x$ and $\hat{x}(x_i) = 0$ for $i \neq m$. Define also the homomorphism $\hat{1} \in D_0 H_j$ by $\hat{1}(1) = 1$. Then $\hat{x} \circ \hat{1}$ belongs to $D_m H_{i+j}$, and we define $x \bar{\circ} 1$ as $[\hat{x} \circ \hat{1}](\omega_m)$, where ω_m is the m^{th} Witt vector.

We can similarly define $1 \bar{\circ} y$ when y is a primitive element of H_j in degree n. We put $1 \bar{\circ} y = [\hat{1} \circ \hat{y}](\omega_n)$.

Since H_i and H_j are both connected Hopf algebras, we know $H = H_i \otimes H_j$ is also a connected Hopf algebra. By [9], this implies that it is connected in the more general sense of having a co-augmentation filtration $\{F_i H\}_{i \geq 0}$ that exhausts it (see Chapter 5). If an element $x \in H$ is in some $F_s H$, then its coproduct is of the form $\Psi_H(x) = 1 \otimes x + x \otimes 1 + \sum x' \otimes x''$, with all x' and x'' in those $F_r H$ with $r < s$. By the exhaustive part in the definition, if $x = a \otimes b \in F_s H$, we can put $a \circ b = \varphi(x)$ to be the unique element in $F_s H_{i+j}$ that has $\Psi_{H_{i+j}}(\varphi(x)) = 1 \otimes \varphi(x) + \varphi(x) \otimes 1 + \sum \varphi(x') \otimes \varphi(x'')$. Thus one defines

$a \circ b = \varphi(x)$ for all $x = a \otimes b \in H$ by induction on the filtration. Since we defined it for the primitives of H, the definition carries over to the whole H and, by construction, φ is a map of coalgebras. $\square$

As before, we denote by $\mathcal{HR}_*$ the category of Hopf rings for graded connected Hopf algebras and by $\mathcal{DR}_*$ the category of graded connected Dieudonné rings.

Consider the functor $D_*^R : \mathcal{HR}_* \to \mathcal{DR}_*$ that takes each sequence of Hopf algebras $\{H_i\}_{i \in \mathbb{Z}}$ to the sequence of Dieudonné modules $\{D_*(H_i)\}_{i \in \mathbb{Z}}$, where D_* is the previous functor $D_* : \mathcal{HA}_* \to \mathcal{DM}_*$, and such that the products $\circ_{ij} : H_i \otimes H_j \to H_{i+j}$ carry over to the products $\circ'_{ij} : D_*H_i \otimes D_*H_j \to D_*H_{i+j}$ as given by Proposition 4.5.

We want to define a functor $U_*^R : \mathcal{DR}_* \to \mathcal{HR}_*$ in a way that makes the pair (D_*^R, U_*^R) an equivalence of categories. We will use the previous functor U_* from Dieudonné *modules* to Hopf *algebras*. For this we first need an auxiliary proposition.

First some notation. We know from previous sections that (D_*, U_*) forms an equivalence of categories between graded connected Hopf algebras and Dieudonné modules. Thus, for each Hopf algebra $H \in \mathcal{HA}_*$, we have $H \cong U_*D_*(H)$. Call φ_H, or simply φ, this isomorphism $\varphi : H \to U_*D_*(H)$. Also, for each Dieudonné module $M \in \mathcal{DM}_*$ there is an isomorphism $\psi : M \to D_*U_*(M)$.

Remark: If M is of the form $M = D_*H$ for some Hopf algebra H, then

$\psi : D_*H \to D_*U_*D_*M$, and the following diagram

$$\begin{tikzcd}
H(n) \arrow[r, "x"] \arrow[dr, "\psi(x)"'] & H \arrow[d, "\varphi"] \\
 & U_*D_*H
\end{tikzcd}$$

defines, for each n, the isomorphism φ in terms of ψ and vice-versa. That is, we can pick φ and ψ in such a way that $\psi(x)(a) = \varphi(x(a))$ for $x \in M = D_*H$ and $a \in H(n)$ (for all n).

Proposition 4.18. *Any bilinear pairing* $\circ_{ij} : M_i \otimes M_j \to M_{i+j}$ *in* $\mathcal{DM}_*$ *induces a bilinear pairing* $\circ'_{ij} : U_*(M_i) \otimes U_*(M_j) \to U_*(M_{i+j})$.

Proof. Given the pairing $M_i \otimes M_j \xrightarrow{\circ} M_{i+j}$ we can get a natural pairing

$$D_*U_*(M_i) \otimes D_*U_*(M_j) \xrightarrow{\hat{\circ}} D_*U_*(M_{i+j})$$

by $\psi(x)\hat{\circ}\psi(y) = \psi(x\circ y)$. This defines also a pairing $U_*(M_i) \otimes U_*(M_j) \xrightarrow{\bar{\hat{\circ}}} U_*(M_{i+j})$ as in the previous lemma, and so we define the pairing $\circ'$ as $\bar{\hat{\circ}}$. $\square$

We can now define the functor $U_*^R : \mathcal{DR}_* \to \mathcal{HR}_*$ as follows:

For each sequence $\{M_i\}_{i\in\mathbb{Z}}$ of Dieudonné modules we get the corresponding sequence $\{U_*(M_i)\}_{i\in\mathbb{Z}}$ of Hopf algebras, where $U_* : \mathcal{DM}_* \to \mathcal{HA}_*$ is the inverse functor to $D_* : \mathcal{HA}_* \to \mathcal{DM}_*$, as given in Theorem 4.3.

Then, given a product $\circ_{ij} : M_i \otimes M_j \to M_{i+j}$, we define the product $\circ'_{ij} : U_*(M_i) \otimes U_*(M_j) \to U_*(M_{i+j})$ as in Proposition 4.5.

(Note that, since U_* and D_* are inverse functors, for each $M \in \mathcal{DM}_*$ we have $M \simeq_{\mathcal{DM}_*} \{\mathrm{Hom}_{\mathcal{HA}_*}(H(n), U_*(M))\}_{n>0}$. Thus each $M \in \mathcal{DM}_*$ is in fact isomorphic to $D_*(H)$ for some Hopf algebra H).

We will need the following auxiliary calculation in the proof of the next Corollary.

Lemma 4.19. *For each primitive x in a Hopf algebra H, we have*

$$\psi^{-1}(\widehat{\varphi(x)}) = \hat{x}$$

Proof. Suppose x is in degree m. We simply follow the calculation:

$$\begin{aligned} \psi^{-1}(\widehat{\varphi(x)})(x_i) &= \varphi^{-1}[\widehat{\varphi(x)}(x_i)] \\ &= \begin{cases} \varphi^{-1}[\varphi(x)] = x, & \text{if } i = m; \\ \varphi^{-1}(0) = 0, & \text{if } i \neq m. \end{cases} \end{aligned}$$

□

Corollary 4.20. *The functor $D^{\mathcal{R}} : \mathcal{HR}_* \to \mathcal{DR}_*$ has a right adjoint $U^{\mathcal{R}} : \mathcal{DR}_* \to \mathcal{HR}_*$ and the pair $(D^{\mathcal{R}}_*, U^{\mathcal{R}}_*)$ forms an equivalence of categories.*

Proof. Since the equivalence between Hopf algebras in $\mathcal{HA}_*$ and Dieudonné modules in $\mathcal{DM}_*$ was already established, at this point we have to deal just with the products $\circ_{ij}$ and $\circ'_{ij}$ defined on these categories.

Suppose we start with the product $\circ_{ij} : H_i \otimes H_j \to H_{i+j}$. This product induces $\circ'_{ij} : D_*H_i \otimes D_*H_j \to D_*H_{i+j}$. This new product $\circ'_{ij}$ induces in its own right a product $\circ''_{ij} : \varphi(H_i) \otimes \varphi(H_j) \to \varphi(H_{i+j})$ where here, as before, φ stands for the isomorphism U_*D_* in the category $\mathcal{HA}_*$.

On the other hand, the product $\circ_{ij} : H_i \otimes H_j \to H_{i+j}$ induces directly a product $\hat{\circ}_{ij} : \varphi(H_i) \otimes \varphi(H_j) \to \varphi(H_{i+j})$ by $\varphi(x) \hat{\circ} \varphi(y) = \varphi(x \circ y)$.

In order to prove the equivalence of ring categories, we need to see that $\hat{\circ} = \circ''$, that is, that the two products defined on $\varphi(H_i) \otimes \varphi(H_j)$ are the same.

Start with $\varphi(x) \in \varphi(H_i)$ and $\varphi(1) \in \varphi(H_j)$, where x is primitive in degree m. Then we have $\varphi(x) \circ'' \varphi(1) = \varphi(x) \bar{\hat{\bar{\circ}}}' \varphi(y)$.

By definition of $\bar{\circ}$ this becomes $[\widehat{\varphi(x)} \hat{\circ}' \widehat{\varphi(1)}](\omega_m)$.

Then, by definition of $\hat{\circ}$, this becomes $[\psi(\psi^{-1}(\widehat{\varphi(x)}) \circ' \psi^{-1}(\widehat{\varphi(1)}))](\omega_m)$.

We also get this to be equal to $\varphi[[\psi^{-1}(\widehat{\varphi}(x)) \circ' \psi^{-1}(\widehat{\varphi}(1))](\omega_m)]$.

Now use the previous Lemma 4.19 and get the last expression equal to $\varphi[\hat{x} \circ' \hat{1}(\omega_m)]$.

But, by the construction of $\circ'$ for Proposition 4.5, this is the same as $\varphi(x \circ 1) = \varphi(x) \hat{\circ} \varphi(1)$, and so the result is proved for this case.

We should also determine that the product of $\varphi(1) \in \varphi(H_i)$ and $\varphi(y) \in \varphi(H_j)$ is the same in both cases whenever y is a primitive element of H_j. This is accomplished in exactly the same fashion we used.

Finally, we should notice that, as seen before, for connected Hopf algebras the map of coalgebras $\circ'' : \varphi(H_i) \otimes \varphi(H_j) \to \varphi(H_{i+j})$ becomes completely determined once we define it on the primitives. Thus the two above products on Hopf algebras are the same for all elements.

To conclude the proof of the equivalence of ring categories, we must do the same reasoning for the two natural products we can have *in Dieudonné modules.* Thus, we proceed as above.

Start with the product $\circ_{ij} : M_i \otimes M_j \to M_{i+j}$.

This product induces $\circ'_{ij} : U_* M_i \otimes U_* M_j \to U_* M_{i+j}$.

This new product $\circ'_{ij}$ induces now a product $\circ''_{ij} : \psi(H_i) \otimes \psi(H_j) \to \psi(H_{i+j})$. Here, ψ is the isomorphism $D_* U_*$ in the category $\mathcal{DM}_*$.

On the other hand, the original product $\circ_{ij} : M_i \otimes M_j \to M_{i+j}$ induces directly a product $\hat{\circ}_{ij} : \psi(M_i) \otimes \psi(M_j) \to \psi(M_{i+j})$ by $\psi(x) \hat{\circ}_{ij} \psi(y) = \psi(x \circ y)$.

We need to show that $\hat{\circ} = \circ''$ as products defined on $\psi(M_i) \otimes \psi(M_j)$.

Start with $\hat{x} \in \psi(M_i) = D_* U_*(M_i) = \{\mathrm{Hom}_{\mathcal{HA}_*}(H(n), U_*(M_i))\}_{n \geq 0}$ in degree $m > 0$ and $\hat{1} \in \Psi(M_j)$ in degree 0, where $x \in U_*(M_i)$ is a primitive in degree m and $1 \in U_*(M_j)$ is in degree zero.

Then, at ω_m, we have

$$[\hat{x} \circ'' \hat{1}](\omega_m) = [\hat{x}(\bar{\hat{\circ}})'\hat{1}](\omega_m) = \hat{x}(x_m) \bar{\hat{\circ}} \hat{1}(1)$$

because of the diagram below defining $(\bar{\hat{\circ}})'$ in terms of $\bar{\hat{\circ}}$.

$$\begin{tikzcd}
 & H(m) \arrow[d] \arrow[dd, bend left=90, looseness=1.5, "\hat{x}(\bar{\hat{\circ}})'\hat{1}"] \\
H(m) \otimes H(0) \arrow[r] \arrow[d, "\hat{x}\otimes\hat{1}"'] & H(m) \boxtimes H(0) \arrow[d, "g"] \\
U(M_i) \otimes U(M_j) \arrow[r, "\bar{\hat{\circ}}"] & U(M_{i+j})
\end{tikzcd}$$

Now we get the previous expression equal to $x \bar{\hat{\circ}} 1$ by definition of $\hat{x}$ and $\hat{1}$. This is $[\hat{x} \hat{\circ} \hat{1}](\omega_m)$ by definition of $\bar{\hat{\circ}}$, and so $\hat{\circ}$ and $\circ''$ do indeed agree at ω_m for $\hat{x}$ and $\hat{1}$.

This implies that they agree on all $x \in H(m)$: First, notice that the map $\iota_m \otimes \iota_0 : H(m) \to H(m) \boxtimes H(0)$ was defined in a way that makes it depend only on its value at x_m (because of commutation with Verschiebungs). Thus, fixing the value of $\hat{x} \circ'' \hat{1}$ at ω_m fixes its value everywhere.

But $\hat{x} \hat{\circ} \hat{1}$ also depends only on its value at x_m, because $\hat{x} \in \Psi(M_i)$ and $\hat{1} \in \Psi(M_j)$, which are in $\mathcal{DR}_*$ and thus satisfy $V(\hat{x} \hat{\circ} \hat{1}) = V(\hat{x}) \circ V(\hat{1})$.

To see that it is enough to show the result in the case of $\hat{x}$ and $\hat{1}$, with x primitive, notice that, by connectedness, $\bar{\hat{\circ}}$ is completely determined by its value on primitives.

Also, because of $V(\hat{x} \hat{\circ} \hat{y}) = V(\hat{x}) \circ V(\hat{y})$, $\hat{\circ}$ is completely determined by its value on primitives.

We should also have determined that the product of $\hat{1} \in \psi(M_i)$ and $\hat{y} \in \psi(M_j)$ is the same in both cases whenever y is a primitive element of H_j. This is accomplished in exactly the same fashion we used. ☐

To deal with the case of group-like in degree zero, we have to construct an inverse functor $U_*^R : \mathcal{DR}_*^o \to \mathcal{HR}_*^o$. For this, we generalize the definitions of induced pairings $\bar{\circ}$ and $\hat{\circ}$ we introduced in the connected case.

(It should be remarked here that the Dieudonné module for a group ring is the underlying $\mathbb{F}_p$-module, concentrated in degree zero and with identity Verschiebung).

Thus, a pairing $\circ : D_*H_i \otimes D_*H_j \to D_*H_{i+j}$ in $\mathcal{DM}_*^o$ induces a pairing $\bar{\circ} : H_i \otimes H_j \to H_{i+j}$ in $\mathcal{HA}_*^o$ by $(ax)\bar{\circ}(by) = (a \circ b)(x \bar{\circ} y)$, where we use the previous definition of $\bar{\circ}$ for connected algebras.

Also, a pairing $\circ : H_i \otimes H_j \to H_{i+j}$ in $\mathcal{HA}_*^o$ induces a pairing $\hat{\circ} : \varphi(H_i) \otimes \varphi(H_j) \to \varphi(H_{i+j})$ in $\mathcal{HA}_*^o$ by $\varphi(ax) \hat{\circ} \varphi(by) = \varphi(a \circ b)(\varphi(x) \hat{\circ} \varphi(y))$, where again we use the previous definition of $\hat{\circ}$ for connected algebras.

Then a pairing $\circ : M_i \otimes M_j \to M_{i+j}$ in $\mathcal{DM}_*^o$ induces a pairing $\circ' : H_i \otimes H_j \to H_{i+j}$ in $\mathcal{HA}_*^o$ by $\circ' = \bar{\hat{\circ}}$.

It becomes just a routine matter to prove that $\circ'' = \hat{\circ}$ on $\varphi(H_i) \otimes \varphi(H_j)$, since the basic work was done already. The same considerations are in order for the two pairings $\circ''$ and $\hat{\circ}$ one can define on $\psi(M_i) \otimes \psi(M_j)$, where M_i and M_j are Dieudonné modules in $\mathcal{DM}_*^o$.

We have thus:

Corollary 4.21. *The functor* $D_*^{\mathcal{R}} : \mathcal{HR}_*^o \to \mathcal{DR}_*^o$ *has a right adjoint* $U_*^{\mathcal{R}} : \mathcal{DR}_*^o \to \mathcal{HR}_*^o$ *and the pair* $(D_*^{\mathcal{R}}, U_*^{\mathcal{R}})$ *forms an equivalence of categories.*

Chapter 5

Dieudonné theory for ungraded Hopf algebras

This chapter will generalize the results from the previous section to some categories of Hopf algebras over $\mathbb{F}_p$ that now don't necessarily carry a grading. We study their Dieudonné modules and Dieudonné rings and prove equivalences between categories of Hopf algebras and Dieudonné modules, and between categories of Hopf rings and Dieudonné rings. We continue to consider that all Hopf algebras are bicommutative. An ungraded Hopf algebra will be called *connected* if it contains an exhaustive coaugmentation filtration. It will be called *geometric-like* if it can be written as $\mathbb{F}_p[G] \otimes H_c$, where $\mathbb{F}_p[G]$ is a group ring and H_c is a connected Hopf algebra. Ungraded connected Hopf algebras are geometric-like.

5.1 Ungraded connected Hopf algebras and their Dieudonné modules

Suppose A is an ungraded cocommutative, coassociative coalgebra with counit over a ring R, together with a coaugmentation $R \to A$. We construct its *coaugmentation filtration* $\{F_qA\}_{q\geq 0}$ by taking the short exact sequence

$$0 \to R \to A \to J(A) \to 0$$

and, using the iterated coproduct on A, defining $F_qA = \ker(A \to J(A)^{\otimes(q+1)})$ for $q \geq 0$.

(In particular, $F_0A \simeq R$ and $F_1A/F_0A \simeq P(A)$.)

Definition 5.1. A coalgebra A as above is *connected* if its coaugmentation filtration exhausts it (that is, if any $x \in A$ is in some F_qA for some $q \geq 0$).

We will denote by $\mathcal{CA}$ the category of connected, cocommutative, coassociative coalgebras with counit over $\mathbb{F}_p$ together with a coaugmentation. We will also call $\mathcal{HA}$ the corresponding category of Hopf algebras; that is, the one formed by the abelian group objects of $\mathcal{CA}$.

We consider the Hopf algebras $H(n) = \mathbb{F}_p[x_1, \cdots, x_n]$ for $n \geq 0$ ($H(0)$ is just $\mathbb{F}_p$). These Hopf algebras are in $\mathcal{HA}$ and moreover they form a basis of projective generators for that category. [2]

We have Hopf algebra maps $\alpha : H(n) \to H(n+1)$ (given by inclusion) and $\bar{V} : H(n+1) \to H(n)$ (defined by $\bar{V}(x_i) = x_{i-1}$ for $i > 0$ and $\bar{V}(x_0) = 0$). $\bar{V}\alpha$ gives the Verschiebung on each $H(n)$.

We have thus a sequence

$$\begin{tikzcd}\cdots \arrow[r] & H(n+1) \arrow[r, "\bar{V}"] & H(n) \arrow[r, "\bar{V}"] & H(n-1) \arrow[r] & \cdots\end{tikzcd}$$

For each Hopf algebra H in $\mathcal{HA}$, this sequence induces a sequence of $\mathbb{F}_p$-modules

$$\begin{tikzcd}\cdots \arrow[r] & \mathrm{Hom}_{\mathcal{HA}}(H(n-1),H) \arrow[r, "\hat{V}"] & \mathrm{Hom}_{\mathcal{HA}}(H(n),H) \arrow[r, "\hat{V}"] & {} \\ & & \mathrm{Hom}_{\mathcal{HA}}(H(n+1),H) \arrow[r] & \cdots\end{tikzcd}$$

where each $\hat{V}$ is given by composition with $\bar{V}$ on the left.

Consider now the $\mathbb{F}_p$-module $DH = \mathrm{colim}_n \mathrm{Hom}_{\mathcal{HA}}(H(n),H)$. Composing on the right with the Verschiebung $v : H \to H$ and the Frobenius $f : H \to H$ gives maps $V : DH \to DH$ and $F : DH \to DH$. Since these maps reflect the Verschiebung and the Frobenius, and since f and v satisfy $fv = vf = p$, we have $FV = VF = p$ as maps from DH to DH. (The same definitions arise from considering composition on the left with the Verschiebung and Frobenius on $\mathbb{F}_p[x_0, x_1, \cdots])$

Definition 5.2. Given a Hopf algebra $H \in \mathcal{HA}$, we define its *Dieudonné module* as the $\mathbb{F}_p$-module $DH = \mathrm{colim}_n \mathrm{Hom}_{\mathcal{HA}}(H(n),H)$ together with the homomorphisms $F : DH \to DH$ and $V : DH \to DH$ given above.

Since any H in $\mathcal{HA}$ is connected, its coaugmentation filtration exhausts it and, moreover, if we write

$$\psi(x) = 1 \otimes x + x \otimes 1 + \sum x' \otimes x''$$

for each $x \in F_q H$, then all the x' and x'' that appear in the expression are in those $F_{q'}H$ that have $q' < q$. Thus, the Verschiebung on such Hopf algebras is eventually zero (that is, for each $x \in H$ we have $v^n x = 0$ for some $n \geq 0$).

This carries over to DH, where we have that for each $x \in DH$ there must exist an $n \geq 0$ such that $V^n x = 0$.

We are ready to define a new category.

Definition 5.3. The category $\mathcal{DM}$ of *ungraded connected Dieudonné modules* has as objects $\mathbb{F}_p$-abelian groups M together with endomorphisms $F : M \to M$ and $V : M \to M$ satisfying $FV = VF = p$ and such that for each $x \in M$ there exists an $n \geq 0$ with $V^n x = 0$.

Similarly to what was done in the graded connected case, we can view a Dieudonné module as a module over the ring $R = \mathbb{Z}_p[V, F]/(FV - p)$.

The considerations we made above give us a functor $D : \mathcal{HA} \to \mathcal{DM}$ that takes a Hopf algebra $H \in \mathcal{HA}$ and produces its Dieudonné module $DH = \mathrm{colim}_n \mathrm{Hom}_{\mathcal{HA}}(H(n), H)$.

Theorem 5.4. *[2] The functor $D : \mathcal{HA} \to \mathcal{DM}$ has a left adjoint $U : \mathcal{DM} \to \mathcal{HA}$, and the pair (D, U) forms an equivalence of categories.*

5.2 Ungraded connected Hopf rings and their Dieudonné rings

Following our previous definition of products $D_*H_i \otimes D_*H_j \to D_*H_{i+j}$ of Dieudonné modules for graded connected Hopf algebras, we want to get similar natural products in the ungraded connected case.

From the remarks made in Chapter 3, there exist tensor products $\boxtimes$ in this category. We will denote them by $\boxtimes_u$ to distinguish them from those in the graded case.

Suppose then that $\circ_{ij} : H_i \otimes H_j \to H_{i+j}$ is a product of ungraded connected Hopf algebras. We want to obtain products

$$DH_i \otimes DH_j \to DH_{i+j}$$

in the corresponding Dieudonné modules. This can be achieved in several steps.

First, consider $x \in \mathrm{Hom}_{\mathcal{HA}}(H(m), H_i)$ and $y \in \mathrm{Hom}_{\mathcal{HA}}(H(n), H_j)$. We will get an element $x \circ y \in DH_{i+j}$. The composition

$$\begin{tikzcd} H(m) \otimes H(n) \arrow[r, "x \otimes y"] & H_i \otimes H_j \arrow[r, "\circ_{ij}"] & H_{i+j} \end{tikzcd}$$

is a bilinear map, so there exists a unique $g : H(m) \boxtimes_u H(n) \to H_{i+j}$ making the following diagram commute.

$$\begin{tikzcd} H(m) \otimes H(n) \arrow[r, "\varphi"] \arrow[d, "x \otimes y"'] & H(m) \boxtimes_u H(n) \arrow[d, "g"] \\ H_i \otimes H_j \arrow[r, "\circ_{ij}"] & H_{i+j} \end{tikzcd}$$

where the top map is the one given in the definition of $\boxtimes_u$.

Notice that this construction is similar to the one done before, only now all maps belong to $\mathcal{HA}$ and not to $\mathcal{HA}_*$.

As before, we consider the map $\eta_m \otimes \eta_n : H(m+n) \to H(m) \boxtimes_u H(n)$ taking x_{m+n} to $\varphi(x_m \otimes x_n)$ and commuting with Verschiebungs.

We can form the composition

$$\begin{tikzcd} H(m+n) \arrow[r, "\iota_m \otimes \iota_n"] & H(m) \boxtimes_u H(n) \arrow[r, "g"] & H_{i+j} \end{tikzcd}$$

We get thus an element $x \circ y$ in $\mathrm{Hom}_{\mathcal{HA}}(H(m+n), H_{i+j})$.

Now suppose $\alpha \in DH_i$ and $\beta \in DH_j$, and write $\alpha = \varphi^m(x)$ for $x \in \mathrm{Hom}_{\mathcal{HA}}(H(m), H_i)$ and $\beta = \varphi^n(y)$ for $y \in \mathrm{Hom}_{\mathcal{HA}}(H(n), H_j)$, where φ^k represents, for each k, the map $\mathrm{Hom}_{\mathcal{HA}}(H(k), H) \to DH$ given by the definition of colimit.

We get, as above, an element $x \circ y \in \mathrm{Hom}_{\mathcal{HA}}(H(m+n), H_{i+j})$, and also an element $\alpha \circ \beta = \varphi^{m+n}(x \circ y) \in DH_{i+j}$.

Proposition 5.5. *The element $\alpha \circ \beta$ does not depend on the x and y picked. That is, if $\varphi^m(x) = \varphi^{m'}(x')$ and $\varphi^n(y) = \varphi^{n'}(y')$ then $\varphi^{m+n}(x \circ y) = \varphi^{m'+n'}(x' \circ y')$*

Proof. Suppose $\varphi^m(x) = \varphi^{m+1}(x')$, where $x \in \mathrm{Hom}_{\mathcal{HA}}(H(m), H_i)$ and $x' \in \mathrm{Hom}_{\mathcal{HA}}(H(m+1), H_j)$.

By definition of colimit we have a commutative diagram

$$\begin{tikzcd}
H(m+1) \arrow[d, "V"'] \arrow[dr, "x'"] & \\
H(m) \arrow[r, "x"] & H_i
\end{tikzcd}$$

We get thus a diagram

$$\begin{tikzcd}
H(m+1) \otimes H(n) \arrow[r, "\varphi_2"] \arrow[d, "V \otimes 1"'] & H(m+1) \boxtimes_u H(n) \arrow[d, "f"] \\
H(m) \otimes H(n) \arrow[r, "\varphi_1"] \arrow[d, "x \otimes y"'] & H(m) \boxtimes_u H(n) \\
H_i \otimes H_j \arrow[r, "\circ_{ij}"] & H_{i+j}
\end{tikzcd}$$

Here, f is the unique map given by the definition of tensor product

$H(m+1) \otimes H(n) \to H(m+1) \boxtimes_u H(n)$.

Then $x \circ y \in \mathrm{Hom}_{\mathcal{HA}}(H(m+n), H_{i+j})$ is the Hopf algebra map completely determined by its value $g\varphi_1(x_m \otimes x_n)$ on x_{m+n} and commuting with Verschiebungs.

Also, $x' \circ y \in \mathrm{Hom}_{\mathcal{HA}}(H(m+n+1), H_{i+j})$ is completely determined by its value $fg\varphi_2(x_{m+1} \otimes x_n)$ on x_{m+n+1}.

We have:

$$\begin{aligned} x' \circ y(x_{m+n+1}) &= fg\varphi_2(x_{m+1} \otimes x_n) \\ &= g\varphi_1(Vx_{m+1} \otimes x_n) \\ &= g\varphi_1(x_m \otimes x_n) \\ &= x \circ y(x_{m+n}) \\ &= x \circ y(Vx_{m+n}) \end{aligned}$$

and so in this case $\varphi^{m+n}(x \circ y) = \varphi^{m+n+1}(x' \circ y)$.

If $\varphi^n(y) = \varphi^{n+1}(y')$, we similarly prove $\varphi^{m+n}(x \circ y) = \varphi^{m+n+1}(x \circ y')$ for any $x \in \mathrm{Hom}_{\mathcal{HA}}(H(m), H_i)$. The general case $\varphi^{m+n}(x \circ y) = \varphi^{m'+n'}(x' \circ y')$ now follows by induction on both $m' - m$ and $n' - n$. $\square$

Corollary 5.6. *Every bilinear pairing* $H_i \otimes H_j \to H_{i+j}$ *in* $\mathcal{HA}$ *induces a pairing* $DH_i \otimes DH_j \to DH_{i+j}$ *in* $\mathcal{DM}$.

As before, we analyze how these pairings relate to V and F defined on the Dieudonné modules.

Proposition 5.7. *Given bilinear pairings* $\circ_{ij} : H_i \otimes H_j \to H_{i+j}$, *the induced pairings* $\circ'_{ij} : DH_i \otimes DH_j \to DH_{i+j}$ *satisfy:*

(a) $V(\alpha \circ \beta) = V\alpha \circ V\beta$*;*

(b) $F\alpha \circ \beta = F(\alpha \circ V\beta)$*;*

(c) $\alpha \circ F\beta = F(V\alpha \circ \beta)$*.*

Proof. We prove the first condition. The others follow similarly.

Suppose $\alpha = \varphi^m(x)$ and $\beta = \varphi^n(y)$.

Then

$$V(\alpha \circ \beta) = V(\varphi^{m+n}(x \circ y)) = \varphi^{m+n}(V(x \circ y))$$

where $V(x \circ y)$ is given by composition on the right with the Verschiebung on H_{i+j}. We then get:

$$\varphi^{m+n}(Vx \circ Vy) = \varphi^m(Vx) \circ \varphi^n(Vy) = V\alpha \circ V\beta$$

where we use the fact that $V(x \circ y) = Vx \circ Vy$ for $x \in \mathrm{Hom}_{\mathcal{HA}}(H(m), H_i)$ and $y \in \mathrm{Hom}_{\mathcal{HA}}(H(n), H_j)$ (The proof of this last fact is exactly the same as in [6] once one ignores the definition of degrees at each step). $\square$

Definition 5.8. A *bilinear pairing* in $\mathcal{DM}$ is a map $f : M_1 \times M_2 \to N$ of abelian groups satisfying $Vf(x,y) = f(Vx,Vy)$, $Ff(x,Vy) = f(Fx,y)$ and $Ff(Vx,y) = f(x,Fy)$.

Proposition 5.7 above stated that a bilinear pairing $\circ : H_1 \otimes H_2 \to K$ in $\mathcal{HA}$ induces a bilinear pairing $\circ' : DH_1 \otimes DH_2 \to DK$ in $\mathcal{DM}$.

Definition 5.9. An *ungraded connected Dieudonné ring* over a perfect field $\mathbb{F}_p$ is a sequence $\{M_i\}_{i \in \mathbb{Z}}$ of ungraded connected Dieudonné modules together with bilinear maps $\circ_{ij} : M_i \otimes M_j \to M_{i+j}$ satisfying the conditions in the previous proposition.

Denote by $\mathcal{HR}$ the category of Hopf rings in the ungraded connected case and by $\mathcal{DR}$ the category of ungraded connected Dieudonné rings.

Now define a functor $D^R : \mathcal{HR} \to \mathcal{DR}$ by considering for each sequence of Hopf algebras $\{H_i\}_{i\in\mathbb{Z}}$ in $\mathcal{HR}$ the sequence of Dieudonné modules $\{D(H_i)\}_{i\in\mathbb{Z}}$, where $D : \mathcal{HA} \to \mathcal{DM}$ is the previously defined functor, and such that the products $\circ_{ij} : H_i \otimes H_j \to H_{i+j}$ carry over to the products $\circ'_{ij} : DH_i \otimes DH_j \to DH_{i+j}$ as before.

To construct a right adjoint U^R for D^R we proceed as we did in the graded case.

We start with a lemma that offers a result symmetric to the one on Corollary 5.6.

Lemma 5.10. *Any bilinear pairing* $\circ_{ij} : DH_i \otimes DH_j \to DH_{i+j}$ *induces a bilinear pairing* $\bar{\circ}_{ij} : H_i \otimes H_j \to H_{i+j}$ *of Hopf algebras.*

Proof. We focus on the primitive elements of $H_i \otimes H_j$.

Given $x \in H_i$ primitive and $1 \in H_j$, pick a positive m and consider $\hat{x} \in \mathrm{Hom}_{\mathcal{HA}}(H(m), H_i)$ (given by $\hat{x}(1) = 1$, $\hat{x}(x_m) = x$ and $\hat{x}(x_i) = 0$ for $i \neq m$). Consider also the homomorphism $\hat{1} \in \mathrm{Hom}_{\mathcal{HA}}(H(0), H_j)$ given by $\hat{1}(1) = 1$. Then we have $\varphi^m(\hat{x}) \circ \varphi^0(\hat{1}) \in DH_{i+j}$, and so $\varphi^m(\hat{x}) \circ \varphi^0(\hat{1}) = \varphi^m(\alpha)$ for some $\alpha \in \mathrm{Hom}_{\mathcal{HA}}(H(m), H_i)$. Call this element $\hat{x} \circ \hat{1}$. Finally, define $x\bar{\circ}1$ as $[\hat{x} \circ \hat{1}](\omega_m)$.

This construction is independent of m: As before, if $\varphi^m(\hat{x}) = \varphi^{m+1}(\widehat{x'})$ then we have $\varphi^m(\hat{x} \circ \hat{1}) = \varphi^{m+1}(\widehat{x'} \circ \hat{1})$, and so $[\hat{x} \circ \hat{1}](x_m) = [\widehat{x'} \circ \hat{1}](x_{m+n})$, proving that the induced pairing is well-defined. (As in the graded case, since we are dealing with connected Hopf algebras, we define the pairing $\bar{\circ} : H_i \otimes H_j \to H_{i+j}$ completely once we have defined it on the primitives of $H_i \otimes H_j$.)

$\square$

We also have that pairings $\circ : H_i \otimes H_j \to H_{i+j}$ induce pairings $\hat{\circ} : UD(H_i) \otimes UD(H_j) \to UD(H_{i+j})$ directly by $UD(x) \hat{\circ} UD(y) = UD(x \circ y)$, and pairings $\circ : M_i \otimes M_j \to M_{i+j}$ induce pairings $\hat{\circ} : DU(M_i) \otimes DU(M_j) \to DU(M_{i+j})$ by $DU(x) \hat{\circ} DU(y) = DU(x \circ y)$.

Finally, pairings $\circ : M_i \otimes M_j \to M_{i+j}$ induce pairings $\circ' : U(M_i) \otimes U(M_j) \to U(M_{i+j})$ given by $\circ' = \bar{\hat{\circ}}$.

Now consider the functor $D^R : \mathcal{HR} \to \mathcal{DR}$ that takes each sequence of Hopf algebras $\{H_i\}_{i \in \mathbb{Z}}$ to the sequence of Dieudonné modules $\{D(H_i)\}_{i \in \mathbb{Z}}$, where D is the previous functor $D : \mathcal{HA} \to \mathcal{DM}$, and such that the products $\circ_{ij} : H_i \otimes H_j \to H_{i+j}$ carry over to the products $\circ'_{ij} : DH_i \otimes DH_j \to DH_{i+j}$ as given by Corollary 5.6.

We want to define a functor $U^R : \mathcal{DR}_* \to \mathcal{HR}_*$ in a way that makes the pair (D^R, U^R) an equivalence of categories. We will use the previous functor U from Dieudonné *modules* to Hopf *algebras*. For this we use the previously defined induced pairings on $U(M_i) \otimes U(M_j)$, for Dieudonné modules M_i and M_j in $\mathcal{DM}$.

For each sequence $\{M_i\}_{i \in \mathbb{Z}}$ of Dieudonné modules we get the corresponding sequence $\{U(M_i)\}_{i \in \mathbb{Z}}$ of Hopf algebras, where $U : \mathcal{DM} \to \mathcal{HA}$ is the inverse functor to $D : \mathcal{HA} \to \mathcal{DM}$, as given in Theorem 5.4.

Then, given a product $\circ_{ij} : M_i \otimes M_j \to M_{i+j}$, we define the product $\circ'_{ij} : U(M_i) \otimes U(M_j) \to U(M_{i+j})$ as above.

Theorem 5.11. *The functor* $D^R : \mathcal{HR} \to \mathcal{DR}$ *has a right adjoint* $U^R : \mathcal{DR} \to \mathcal{HR}$ *and the pair* (D^R, U^R) *forms an equivalence of categories.*

Proof. To prove the equivalence of ring categories, we first show that the products $\hat{\circ}$ and $\circ''$ are the same on $UD(H_i) \otimes UD(H_j)$, for Hopf algebras H_i and H_j in $\mathcal{HA}$. Most of the work follows directly from what was done in the graded case:

Consider $UD(x) \in UD(H_i)$ and $UD(1) \in UD(H_j)$. Pick an $m > 0$ and notice we have an element $\widehat{UD(x)} \in \mathrm{Hom}_{\mathcal{HA}}(H(m), UD(H_i))$. Also, there is an element $\widehat{UD(1)} \in \mathrm{Hom}_{\mathcal{HA}}(H(0), UD(H_j))$.

Then $UD(x) \circ'' UD(1) = UD(x) \bar{\hat{\circ}}' UD(1) = \widehat{UD(x)} \hat{\circ}' \widehat{UD(1)}(\omega_m)$, where the element $\widehat{UD(x)} \hat{\circ}' \widehat{UD(1)} \in \mathrm{Hom}_{\mathcal{HA}}(H(m), H_{i+j})$ was defined uniquely by $\varphi^m(\widehat{UD(x)} \hat{\circ}' \widehat{UD(1)}) = \varphi^m(\widehat{UD(x)}) \hat{\circ}' \varphi^0(\widehat{UD(1)})$.
The last expression becomes $DU[(DU)^{-1}(\varphi^m(\widehat{UD(x)})) \circ' (DU)^{-1}(\varphi^0(\widehat{UD(1)}))] = DU[\varphi^m(\hat{x}) \circ' \varphi^0(\hat{1})] = DU[\varphi^m(\hat{x} \circ' \hat{1})] = \varphi^m(\alpha)$,
where $\alpha \in \mathrm{Hom}_{\mathcal{HA}}(H(m), UD(H_{i+j}))$ satisfies $\alpha(\omega_m) = UD(\hat{x} \circ' \hat{1}(\omega_m)) = UD(x \circ 1)$, and so indeed $\widehat{UD(x)} \hat{\circ}' \widehat{UD(1)}(\omega_m) = UD(x \circ 1)$, making $UD(x) \circ'' UD(1) = UD(x) \hat{\circ} UD(1)$ as we wanted to show.

The same remarks we made for the graded connected case imply that this is enough to show $\circ'' = \hat{\circ}$ on the whole $UD(H_i) \otimes UD(H_j)$.

To conclude the proof of the equivalence of ring categories, we must do the same reasoning for the two natural products we can have *in Dieudonné modules.* Thus, we proceed as above:

Start with the product $\circ_{ij} : M_i \otimes M_j \to M_{i+j}$. This product induces $\circ'_{ij} = \bar{\hat{\circ}} : UM_i \otimes UM_j \to UM_{i+j}$ in Hopf algebras, which induces $\circ''_{ij} : DUM_i \otimes DUM_j \to DUM_{i+j}$.

The original product $\circ_{ij} : M_i \otimes M_j \to M_{i+j}$ also induces directly $\hat{\circ} : DU(M_i) \otimes DU(M_j) \to DU(M_{i+j})$ by $DU(x) \hat{\circ} DU(y) = DU(x \circ y)$.

We show that $\hat{\circ} = \circ''$ as products on $DU(M_i) \otimes DU(M_j)$.

We have $\varphi^m(\alpha)\hat{\circ}\varphi^n(\beta) = \varphi^{m+n}(\alpha\hat{\circ}\beta)$, where $\alpha \in \mathrm{Hom}_{\mathcal{HA}}(H(m), U(M_i))$ and $\beta \in \mathrm{Hom}_{\mathcal{HA}}(H(n), U(M_j))$ (Here, $\alpha\hat{\circ}\beta$ is uniquely determined by the expression above).

But $\alpha\hat{\circ}\beta = \alpha \circ'' \beta$ because of Chapter 4, where $\circ''$ should be interpreted as in the graded connected case, only with $\boxtimes_u$ substituted in the diagrams.

We get $\varphi^{m+n}(\alpha \circ'' \beta) = \varphi^m(\alpha) \circ'' \varphi^n(\beta)$, and so $\hat{\circ} = \circ''$. □

5.3 Dieudonné theory for ungraded geometric-like Hopf algebras

In this section we generalize the results in Sections 4.3 and 4.4 in order to accomodate the case of ungraded geometric-like Hopf algebras.

Suppose H is an ungraded Hopf algebra over a commutative ring k with unit and consider $X(H) = \mathrm{Hom}_{\mathcal{CA}}(k, H)$. Define H_o as the group ring $k[X(H)]$.

(Call the elements in this group ring the *group-like elements of* H). Define also $H_c = k \otimes_{H_o} H$. Then H_c is an ungraded connected Hopf algebra over k.

We will consider in this section those Hopf algebras H that are isomorphic to $H_o \otimes H_c$.

Definition 5.12. An ungraded Hopf algebra H over k is *geometric-like* if it can be written as $H \cong H_o \otimes_k H_c$, where H_o is a group ring over k and H_c is an ungraded connected Hopf algebra over k.

Definition 5.13. Call $\mathcal{HA}^o$ the category of bicommutative Hopf algebras over $\mathbb{F}_p$ that are geometric-like.

Now define $H(0) = \mathbb{F}_p[\mathbb{Z}]$ and remember that, for $n > 0$, $H(n)$ was defined as $H(n) = \mathbb{F}_p[x_1, \cdots, x_n]$.

For a Hopf algebra H in $\mathcal{HA}^o$, define $DH = \operatorname{colim}\{D_n H = \operatorname{Hom}_{\mathcal{HA}^o}(H(n), H)\}_{n \geq 0}$.

Composing on the right with the Verschiebung $v : H \to H$ and the Frobenius $f : H \to H$ gives maps $V : DH \to DH$ and $F : DH \to DH$. As before, if $m > 0$ we have $FV(\varphi^m(\alpha)) = VF(\varphi^m(\alpha)) = p\varphi^m(\alpha)$ for $\alpha \in \operatorname{Hom}_{\mathcal{HA}}(H(m), H)$. If $m = 0$ we have $V\varphi^0(\alpha) = \varphi^0(\alpha)$ for $\alpha \in \operatorname{Hom}_{\mathcal{HA}}(H(0), H)$ since the Verschiebung on group-like elements of H is the identity.

We can thus define a new category.

Definition 5.14. The category $\mathcal{DM}^o$ has as objects abelian groups $M = M_o \otimes M_c$ together with maps $V : M \to M$ and $F : M \to M$ satisfying

(a) M_c is a Dieudonné module in $\mathcal{DM}$;

(b) $VF = FV = p$ on $1 \otimes M_c$;

(c) V is the identity on $M_o \otimes 1$.

As before, we have:

Theorem 5.15. *The functor $D : \mathcal{HA}^o \to \mathcal{DM}^o$ gives an equivalence of categories.*

We next want to deal with ring objects.

First, notice that in $\mathcal{HA}^o$ products are defined as

$$\tilde{\varphi} : (A_o \otimes A_c) \otimes (B_o \otimes B_c) \to (C_o \otimes C_c)$$

with $\tilde{\varphi}(ax \otimes by) = \varphi(a \otimes b)\varphi'(x \otimes y)$, where $\varphi : A_o \otimes B_o \to C_o$ is a product of group rings and $\varphi' : A_c \otimes B_c \to C_c$ is a pairing of graded connected Hopf algebras. (Here we write ax for $a \otimes x \in A_o \otimes A_c$, etc.)

A bilinear map in $\mathcal{DM}^o$ will be a map $\tilde{\varphi} : (M_o \otimes M_c) \otimes (N_o \otimes N_c) \to (K_o \otimes K_c)$ given by $\tilde{\varphi}(am \otimes bm) = \varphi(a \otimes b)\varphi'(m \otimes n)$, where $\varphi : M_o \otimes N_o \to K_o$ is a product in group rings and $\varphi' : M \otimes N \to K$ is a bilinear map in $\mathcal{DM}$ as defined before.

Following the reasoning from Section 4.3, any pairing $\circ_{ij} : H_i \otimes H_j \to H_{i+j}$ in $\mathcal{HA}^o$ will induce a pairing $\circ'_{ij} : DH_i \otimes DH_j \to DH_{i+j}$ in $\mathcal{DM}^o$. As before, the diagram

$$\begin{tikzcd}
H(0) \otimes H(0) \arrow[r] \arrow[d, "a\otimes b"'] & H(0) \boxtimes_u H(0) \arrow[d, "g"] \\
(H_i)_o \otimes (H_j)_o \arrow[r, "\circ"] & (H_{i+j})_o
\end{tikzcd}$$

allows us to define products $\circ_{ij} : (DH_i)_o \otimes (DH_j)_o \to (DH_{i+j})_o$. We also have the induced products on the Dieudonné modules for the connected parts of the Hopf algebras H, given by the restriction to $\mathcal{HA}$ of the pairing $\circ$. Thus we define

$$ax \circ' by = (a \circ b)(x \circ' y)$$

We will denote by $\mathcal{HR}^o$ the category of ungraded geometric-like Hopf rings (that is, of the ring objects for $\mathcal{HA}^o$) and by $\mathcal{DR}^o$ the category of Dieudonné rings in this case.

Consider now the functor $D^R : \mathcal{HR}^o \to \mathcal{DR}^o$ that takes each sequence of Hopf algebras $\{H_i\}_{i \in \mathbb{Z}}$ to the sequence of Dieudonné modules $\{D(H_i)\}_{i \in \mathbb{Z}}$, where D is the previous functor $D : \mathcal{HA}^o \to \mathcal{DM}^o$, and such that the products $\circ_{ij} : H_i \otimes H_j \to H_{i+j}$ carry over to the products $\circ'_{ij} : DH_i \otimes DH_j \to DH_{i+j}$ as given in the preceding paragraph.

To construct an inverse functor $U^R : \mathcal{DR}^o \to \mathcal{HR}^o$, we again generalize the definitions of induced pairings $\bar{\circ}$ and $\hat{\circ}$ introduced in the connected case. A pairing $\circ : D_*H_i \otimes D_*H_j \to D_*H_{i+j}$ in $\mathcal{DM}^o$ induces a pairing $\bar{\circ} : H_i \otimes H_j \to H_{i+j}$ in $\mathcal{HA}^o$ by $(ax)\bar{\circ}(by) = (a \circ b)(x \bar{\circ} y)$, where we use the previous definition of $\bar{\circ}$ for connected algebras.

Also, a pairing $\circ : H_i \otimes H_j \to H_{i+j}$ in $\mathcal{HA}^o$ induces a pairing $\hat{\circ} : \varphi(H_i) \otimes \varphi(H_j) \to \varphi(H_{i+j})$ in $\mathcal{HA}^o$ by $\varphi(ax)\hat{\circ}\varphi(by) = \varphi(a \circ b)(\varphi(x)\hat{\circ}\varphi(y))$, using the previous definition of $\hat{\circ}$ for connected algebras.

Then a pairing $\circ : M_i \otimes M_j \to M_{i+j}$ in $\mathcal{DM}^o$ induces a pairing $\circ' : H_i \otimes H_j \to H_{i+j}$ in $\mathcal{HA}^o$ by $\circ' = \bar{\hat{\circ}}$.

The proof of the next Corollary follows directly from the work we did in the connected case in Section 5.2 and the methods used in the graded case generalizations of Section 4.4.

Corollary 5.16. *The functor* $D^R : \mathcal{HR}^o \to \mathcal{DR}^o$ *has a right adjoint* $U^R : \mathcal{DR}^o \to \mathcal{HR}^o$ *and the pair* (D^R, U^R) *forms an equivalence of categories.*

Chapter 6

Dieudonné theory for periodically graded Hopf algebras

In this chapter we consider periodically graded bicommutative Hopf algebras over $\mathbb{F}_p$ and, as before, we study their corresponding Dieudonné modules, plus the corresponding Dieudonné rings for the Hopf rings. We again focus on connected and geometric-like Hopf algebras, with definitions similar to those we presented in the two previous chapters.

6.1 Periodically graded connected Hopf algebras and their Dieudonné modules

The basic material in this section can be found in [13].

We will have to restrict ourselves to a special grading if we want the maps we define to be in the category of graded Hopf algebras. For this, start by fixing an $n > 0$. We will write $m = p^n - 1$ and consider $2m$-graded bicommutative Hopf algebras over $\mathbb{F}_p$, that is, those that are graded over $\mathbb{Z}/2(p^n - 1)$, and will focus on those Hopf algebras that are concentrated in even degrees (having elements in odd degree implies, by commutativity, that the Hopf algebra has elements squaring to zero; in fact, the category of $2m$-graded bicommutative Hopf algebras splits as a direct product of the category of those concentrated in even degrees and a category of primitively generated exterior algebras - see [7]).

A $2m$-graded Hopf algebra concentrated in even degrees will be called *connected* if $H_o \cong \mathbb{F}_p$. We should notice that any such Hopf algebra can be viewed as an *ungraded* Hopf algebra and that being connected as a $2m$-graded Hopf algebra implies that it is connected in the general sense of ungraded Hopf algebras (that is, having an exhaustive coaugmentation filtration).

Definition 6.1. We will denote by $\mathcal{HA}_m$ the category whose objects are $2m$-graded connected bicommutative Hopf algebras over $\mathbb{F}_p$ concentrated in even degrees.

Since Hopf algebras in $\mathcal{HA}_m$ can be viewed as ungraded connected ones, the definition of Dieudonné modules from Section 5.1 carries over to this section and we have $DH = \text{colim}\{\text{Hom}_{\mathcal{HA}}(H(n), H)\}$ for any $H \in \mathcal{HA}_m$.

In this case, though, we want a new definition that takes care of the grading, as the maps $H(n) \to H$ in the colimit above are not necessarily in $\mathcal{HA}_m$ (in fact, we haven't even put a grading on $H(n)$ yet).

We follow what was done in the graded connected case before. Start by giving each indeterminate x_i degree $2p^i t \mod 2m$ for a fixed t satisfying $1 \leq t \leq p^n - 1$ and define $H(s,t) = \mathbb{F}_p[x_0, x_1, \cdots, x_s]$ for each s, giving it the unique Hopf algebra structure graded over $2m$ that makes the Witt polynomials primitive. The Verschiebung on these $H(s,t)$ is given by $V(x_i) = x_{i-1}$ for $i > 0$.

The Hopf algebra maps $\bar{V} : H(s+1,t) \to H(s,t)$ corresponding to the ones used in the beginning of Section 5.1 to define the colimit are not necessarily in $\mathcal{HA}_m$, but we can choose some iterate of $\bar{V}$ that does preserve degree: in fact, since the Verschiebung divides degree by p and our grading imposes $p^n = 1$, we have that $\bar{V}^n$ is a map in $\mathcal{HA}_m$ and so we can define Dieudonné modules for $H \in \mathcal{HA}_m$ by:

$$D_t H = \operatorname{colim}\{\operatorname{Hom}_{\mathcal{HA}_m}(H(s,t), H)\}$$

Notice that each $\bar{V}^n$ is a map from a $H(s,t)$ with $s > n$ to $H(s-n,t)$. This means that, by definition of colimit, an element $\alpha \in \operatorname{Hom}_{\mathcal{HA}_m}(H(s,t), H)\}$ whose image under the colimit map is in DH is completely determined by the restriction of that map to $H(s',t)$, where $0 \leq s' < n$ and $s' = s \pmod{n}$.

If we now vary t we get a collection $\{D_t\}_{1 \leq t \leq p^n - 1}$ (one should notice that, since $p^n \equiv 1 \pmod{m}$ in our grading, the modules $D_t H$ for $t > p^n - 1$ would start repeating the ones already determined, and so $DH = \{D_t H\}_{1 \leq t \leq p^n - 1}$ is a m-graded module).

Next we define the homomorphisms V and F on DH.

For each s and t, the map $H(s,pt) \to H(s,t)$ that takes x_i to ${x_i}^p$ preserves degree, and so is a homomorphism in $\mathcal{HA}_m$. The following diagram is commutative because the composition of the inclusion $H(s,t) \to H(s,pt)$ with the above map gives the Frobenius F on $H(s,t)$ and the Frobenius commutes with the Verschiebung.

$$\begin{tikzcd}
H(0,t) & H(n,t) \arrow[l] & \cdots \arrow[l] & H(n(s-1),t) \arrow[l] & H(ns,t) \arrow[l] & \cdots \arrow[l] \\
H(0,pt) \arrow[u] & H(n,pt) \arrow[l] \arrow[u] & \cdots \arrow[l] & H(n(s-1),t) \arrow[l] \arrow[u] & H(ns,pt) \arrow[l] \arrow[u] & \cdots \arrow[l]
\end{tikzcd}$$

This diagram allows us to define the homomorphism $V : D_tH \to D_{pt}H$.

Similarly, the composition $H(s,t/p) \to H(s-1,t) \to H(s,t)$ taking x_i to x_{i-1} preserves degree, and so the diagram

$$\begin{tikzcd}
H(0,t) & H(n,t) \arrow[l] & \cdots \arrow[l] & H(n(s-1),t) \arrow[l] & H(ns,t) \arrow[l] & \cdots \arrow[l] \\
H(0,t/p) \arrow[u] & H(n,t/p) \arrow[l] \arrow[u] & \cdots \arrow[l] \arrow[u] & H(n(s-1),t/p) \arrow[l] \arrow[u] & H(ns,t/p) \arrow[l] \arrow[u] & \cdots \arrow[l]
\end{tikzcd}$$

allows us to define a homomorphism $V : D_tH \to D_{p/t}H$.

We can give a new definition.

Definition 6.2. The *Dieudonné module* for a Hopf algebra $H \in \mathcal{HA}_m$ is the m-graded abelian group

$$DH = \{D_tH = \operatorname{colim}\{\operatorname{Hom}_{\mathcal{HA}_m}(H(s,t), H)\}\}$$

together with the above homomorphisms

$$F : D_tH \to D_{pt}H$$

and

$$V : D_{pt}H \to D_tH$$

We also define a category of Dieudonné modules.

Definition 6.3. We call $\mathcal{DM}_m$ the category of m-graded modules M over $\mathbb{F}_p$ together with maps V and F satisfying $FV = VF = p$ and such that for each $x \in M$ there exists an $r \geq 1$ with $V^r(x) = 0$

The equivalence between Hopf algebras and Dieudonné modules becomes, in this case:

Theorem 6.4. *[13] The above map D is a functor $D : \mathcal{HA}_m \to \mathcal{DM}_m$ that has a right adjoint $U : \mathcal{D}_m \to \mathcal{HA}_m$, and the pair (D, U) forms an equivalence of categories.*

Again, the proof confirms the fact that an abelian category with a set of small projective generators is equivalent to a category of modules over some ring [3] [4].

6.2 Periodically graded connected Hopf rings and their Dieudonné rings

Denote by $\mathcal{HR}_m$ the category of Hopf rings in the $2m$-graded connected case, that is, whose objects are collections $\{H_i\}$ of $2m$-graded connected Hopf algebras (concentrated in even degrees) together with maps $H_i \otimes H_j \to H_{i+j}$ of $2m$-graded coalgebras.

We want to show this category is equivalent to a category of Dieudonné rings.

Proposition 6.5. *Every bilinear pairing $H_i \otimes H_j \to H_{i+j}$ in $\mathcal{HA}_m$ induces a pairing $DH_i \otimes DH_j \to DH_{i+j}$ in $\mathcal{DM}_m$.*

Proof. Suppose that $\circ_{ij} : H_i \otimes H_j \to H_{i+j}$ is a product of $2m$-graded connected Hopf algebras. We want to obtain products

$$DH_i \otimes DH_j \to DH_{i+j}$$

in the corresponding Dieudonné modules.

Consider then $x \in D_r H_i$ and $y \in D_s H_j$. As we said above, we can consider x to be an element in $\mathrm{Hom}_{\mathcal{HA}_m}(H(n,r), H_i)$ (it is completely determined by its values on $x_0, \cdots, x_n$). Similarly $y \in \mathrm{Hom}_{\mathcal{HA}_m}(H(n,s), H_j)$. We can get a map in $\mathrm{Hom}_{\mathcal{HA}_m}$ using a diagram similar to the ones we used before:

$$\begin{tikzcd}
H(n,r) \otimes H(n,s) \arrow[r] \arrow[d, "x\otimes y"'] & H(n,r) \boxtimes_m H(n,s) \arrow[d, "g"] \\
H_i \otimes H_j \arrow[r, "\circ_{ij}"] & H_{i+j}
\end{tikzcd}$$

Here, as before, $\boxtimes_m$ is the tensor product in our case. We have a map $\eta_r \otimes \eta_s : H(n, r+s) \to H(n,r) \boxtimes_m H(n,s)$ and can form the composition

$$\begin{tikzcd}
H(n,r+s) \arrow[r, "\iota_r\otimes\iota_s"] & H(n,r) \boxtimes_m H(n,s) \arrow[r, "g"] & H_{i+j}
\end{tikzcd}$$

This element completely determines a map in $\mathrm{Hom}_{\mathcal{HA}_m}(H(n,r+s), H_{i+j})$, and this map defines an $x \circ y \in D_{r+s} H_{i+j}$.

□

As before, we analyze how these pairings relate to V and F defined on the Dieudonné modules.

Proposition 6.6. *Given bilinear pairings* $\circ_{ij} : H_i \otimes H_j \to H_{i+j}$, *the induced pairings* $\circ'_{ij} : DH_i \otimes DH_j \to DH_{i+j}$ *satisfy:*

(a) $V(x \circ y) = Vx \circ Vy$;

(b) $Fx \circ y = F(x \circ Vy)$;

(c) $x \circ Fy = F(Vy \circ y)$.

Proof. Consider the first condition. The others follow similarly.

Suppose $x \in \mathrm{Hom}_{\mathcal{HA}_m}(H(n,r), H_i)$ and $y \in \mathrm{Hom}_{\mathcal{HA}_m}(H(n,s), H_j)$.

Then $x \circ y$ is given by the following diagram.

$$\begin{tikzcd}
 & H(n, r+s) \arrow[d] \\
H(n,r) \otimes H(n,s) \arrow[r] \arrow[d, "x \otimes y"'] & H(n,r) \boxtimes_m H(n,s) \arrow[d, "g"] \\
H_i \otimes H_j \arrow[r, "\circ_{ij}"] & H_{i+j}
\end{tikzcd}$$

The element Vx can be seen to be given by the composition $H(n, r/p) \to H(n,r) \to H_i$, where the left map takes x_i to x_{i-1} (this definition agrees with the previous one since the colimit defining $x \in D_r H_i$ is completely defined by a map $H(n,r) \to H_i$).

At the same time, the element Vy is given by the composition $H(n, s/p) \to H(n,s) \to H_j$.

There exists a unique map $H(n, r/p) \boxtimes_m H(n, s/p) \to H(n,r) \boxtimes_m H(n,s)$ (because of the definition of tensor products). Putting this map into the following diagram defining $Vx \circ Vy$ proves the result.

$$\begin{tikzcd}
 & & H(n,(r+s)/p) \arrow[dl] \arrow[dr] & \\
H(n,r/p) \otimes H(n,s/p) \arrow[r] \arrow[d] & H(n,r/p) \boxtimes_m H(n,s/p) \arrow[dr] & & H(n,r+s) \arrow[dl] \arrow[ddl] \\
H(n,r) \otimes H(n,s) \arrow[rr] \arrow[d] & & H(n,r) \boxtimes_m H(n,s) \arrow[d] & \\
H_i \otimes H_j \arrow[rr] & & H_{i+j} &
\end{tikzcd}$$

□

Definition 6.7. A *bilinear pairing* in $\mathcal{DM}_m$ is a map $f : M_1 \times M_2 \to N$ of abelian groups satisfying $Vf(x,y) = f(Vx,Vy)$, $Ff(x,Vy) = f(Fx,y)$ and $Ff(Vx,y) = f(x,Fy)$.

We just saw that a bilinear pairing $\circ : H_1 \otimes H_2 \to K$ in $\mathcal{HA}_m$ induces a bilinear pairing $\circ' : DH_1 \otimes DH_2 \to DK$ in $\mathcal{DM}_m$.

Definition 6.8. An *m-graded connected Dieudonné ring* over a perfect field $\mathbb{F}_p$ is a sequence $\{M_i\}_{i\in\mathbb{Z}}$ of m-graded connected Dieudonné modules together with bilinear maps $\circ_{ij} : M_i \otimes M_j \to M_{i+j}$ satisfying the conditions in the previous proposition.

Denote by $\mathcal{HR}_m$ the category of Hopf rings in the $2m$-graded connected case and by $\mathcal{DR}_m$ the category of m-graded connected Dieudonné rings.

Define a functor $D^R : \mathcal{HR}_m \to \mathcal{DR}_m$ by considering for each sequence of Hopf algebras $\{H_i\}_{i\in\mathbb{Z}}$ in $\mathcal{HR}_m$ the sequence of Dieudonné modules $\{D(H_i)\}_{i\in\mathbb{Z}}$, where $D : \mathcal{HA}_m \to \mathcal{DM}_m$ is the previously defined functor, and such that the products $\circ_{ij} : H_i \otimes H_j \to H_{i+j}$ carry over to the products $\circ'_{ij} : DH_i \otimes DH_j \to DH_{i+j}$ as before.

To construct a right adjoint U^R for D^R, we extend the results in the previous chapters.

Lemma 6.9. *Any bilinear pairing* $\circ_{ij} : DH_i \otimes DH_j \to DH_{i+j}$ *induces a bilinear pairing* $\bar{\circ}_{ij} : H_i \otimes H_j \to H_{i+j}$ *of Hopf algebras.*

Proof. The proof follows as in the graded and ungraded cases.

Given $x \in H_i$ primitive and $1 \in H_j$, consider $\hat{x} \in \mathrm{Hom}_{\mathcal{HA}_m}(H(n,r), H_i)$ (given by $\hat{x}(1) = 1$, $\hat{x}(x_n) = x$ and $\hat{x}(x_i) = 0$ for $i \neq n$). Consider also the homomorphism $\hat{1} \in \mathrm{Hom}_{\mathcal{HA}_m}(H(n,0), H_j)$ given by $\hat{1}(1) = 1$. Then we define $x \bar{\circ} 1 = [\hat{x} \circ \hat{1}](\omega_n)$

□

We also have that pairings $\circ : H_i \otimes H_j \to H_{i+j}$ induce pairings $\hat{\circ} : UD(H_i) \otimes UD(H_j) \to UD(H_{i+j})$ directly by $UD(x) \hat{\circ} UD(y) = UD(x \circ y)$, and pairings $\circ : M_i \otimes M_j \to M_{i+j}$ induce pairings $\hat{\circ} : DU(M_i) \otimes DU(M_j) \to DU(M_{i+j})$ by $DU(x) \hat{\circ} DU(y) = DU(x \circ y)$.

Finally, pairings $\circ : M_i \otimes M_j \to M_{i+j}$ induce pairings $\circ' : U(M_i) \otimes U(M_j) \to U(M_{i+j})$ given by $\circ' = \bar{\hat{\circ}}$.

Now consider the functor $D^R : \mathcal{HR}_m \to \mathcal{DR}_m$ that takes each sequence of Hopf algebras $\{H_i\}_{i\in\mathbb{Z}}$ to the sequence of Dieudonné modules $\{D(H_i)\}_{i\in\mathbb{Z}}$, where D is the previous functor $D : \mathcal{HA}_m \to \mathcal{DM}_m$, and such that the products $\circ_{ij} : H_i \otimes H_j \to H_{i+j}$ carry over to the products $\circ'_{ij} : DH_i \otimes DH_j \to DH_{i+j}$ as given above.

Define a functor $U^R : \mathcal{DR}_m \to \mathcal{HR}_m$ by assigning to each sequence $\{M_i\}_{i\in\mathbb{Z}}$ of Dieudonné modules the corresponding sequence $\{U(M_i)\}_{i\in\mathbb{Z}}$ of Hopf algebras, where $U : \mathcal{DM}_m \to \mathcal{HA}_m$ is the inverse functor to $D : \mathcal{HA}_m \to \mathcal{DM}_m$.

Then, given a product $\circ_{ij} : M_i \otimes M_j \to M_{i+j}$, we define the product $\circ'_{ij} : U(M_i) \otimes U(M_j) \to U(M_{i+j})$ as above.

As in the previous chapters, we have:

Theorem 6.10. *The functor $D^R : \mathcal{HR}_m \to \mathcal{DR}_m$ has a right adjoint $U^R : \mathcal{DR}_m \to \mathcal{HR}_m$ and the pair (D^R, U^R) forms an equivalence of categories.*

6.3 Dieudonné theory for periodically graded geometric-like Hopf algebras

This section contains the generalization of the results in the previous sections in order to accomodate the case of $2m$-graded geometric-like Hopf algebras.

Following Chapter 5, we will call *geometric-like m-graded Hopf algebras* those that can be written as $H \cong H_o \otimes_k H_c$, where H_o is a group ring over k and H_c is a $2m$-graded connected Hopf algebra over k (concentrated in even degrees).

Definition 6.11. Call $\mathcal{HA}^o_m$ the category of bicommutative Hopf algebras over $\mathbb{F}_p$ that are geometric-like.

Now define $H(0,s) = \mathbb{F}_p[\mathbb{Z}]$ for any s and remember that, for $n > 0$, $H(n,s)$ was defined as $H(n,s) = \mathbb{F}_p[x_0, \cdots, x_n]$, with each x_i in degree $2p^i s$.

For a Hopf algebra H in $\mathcal{HA}_m^o$, define $D_o H = \mathrm{Hom}_{\mathcal{HA}_m^o}(H(0,s), H)$ and $D_t H$ (for $t > 0$) as before.

We have that the Verschiebung on $H(0,s)$ is the identity, and so we can extend V on Dieudonné modules by declaring it to be the identity on D_o. Also, define F on D_o in a way that preserves $FV = VF = p$.

We can thus define a new category.

Definition 6.12. The category $\mathcal{DM}_m^o$ has as objects abelian groups $M = M_o \otimes M_c$ together with maps $V : M \to M$ and $F : M \to M$ satisfying

(a) M_c is a Dieudonné module in $\mathcal{DM}_m$;

(b) $VF = FV = p$ on $1 \otimes M_c$;

(c) V is the identity on $M_o \otimes 1$.

Theorem 6.13. *The functor* $D : \mathcal{HA}_m^o \to \mathcal{DM}_m^o$ *gives an equivalence of categories.*

We next want to deal with ring objects.

A bilinear map in $\mathcal{DM}_m^o$ will be a map $\tilde{\varphi} : (M_o \otimes M_c) \otimes (N_o \otimes N_c) \to (K_o \otimes K_c)$ given by $\tilde{\varphi}(ax \otimes by) = \varphi(a \otimes b)\varphi'(x \otimes y)$, where $\varphi : M_o \otimes N_o \to K_o$ is a product in group rings and $\varphi' : M \otimes N \to K$ is a bilinear map in $\mathcal{DM}_m$ as defined before.

Any pairing $\circ_{ij} : H_i \otimes H_j \to H_{i+j}$ in $\mathcal{HA}_m^o$ induces a pairing

$\circ'_{ij} : DH_i \otimes DH_j \to DH_{i+j}$ in $\mathcal{DM}^o_m$ (same reasoning as for the previous two cases).

We will denote by $\mathcal{HR}^o_m$ the category of $2m$-graded geometric-like Hopf rings (that is, of the ring objects for $\mathcal{HA}^o_m$) and by $\mathcal{DR}^o_m$ the category of Dieudonné rings in this case.

There is a functor $D^R : \mathcal{HR}^o_m \to \mathcal{DR}^o_m$ taking each sequence of Hopf algebras $\{H_i\}_{i\in\mathbb{Z}}$ to the sequence of Dieudonné modules $\{D(H_i)\}_{i\in\mathbb{Z}}$, where D is the previous functor $D : \mathcal{HA}^o_m \to \mathcal{DM}^o_m$, and such that the products $\circ_{ij} : H_i \otimes H_j \to H_{i+j}$ carry over to the products ${\circ'}_{ij} : DH_i \otimes DH_j \to DH_{i+j}$.

To construct an inverse functor $U^R : \mathcal{DR}^o \to \mathcal{HR}^o$, we again generalize the definitions of induced pairings $\bar{\circ}$ and $\hat{\circ}$ introduced in the connected case.

A pairing $\circ : D_*H_i \otimes D_*H_j \to D_*H_{i+j}$ in $\mathcal{DM}^o_m$ induces a pairing $\bar{\circ} : H_i \otimes H_j \to H_{i+j}$ in $\mathcal{HA}^o_m$ by $(ax)\bar{\circ}(by) = (a \circ b)(x\bar{\circ}y)$, where we use the previous definition of $\bar{\circ}$ for connected algebras.

Also, a pairing $\circ : H_i \otimes H_j \to H_{i+j}$ in $\mathcal{HA}^o_m$ induces a pairing $\hat{\circ} : \varphi(H_i) \otimes \varphi(H_j) \to \varphi(H_{i+j})$ in $\mathcal{HA}^o_m$ by $\varphi(ax)\hat{\circ}\varphi(by) = \varphi(a \circ b)(\varphi(x)\hat{\circ}\varphi(y))$, using the previous definition of $\hat{\circ}$ for connected algebras.

Then a pairing $\circ : M_i \otimes M_j \to M_{i+j}$ in $\mathcal{DM}^o_m$ induces a pairing $\circ' : H_i \otimes H_j \to H_{i+j}$ in $\mathcal{HA}^o_m$ by $\circ' = \bar{\hat{\circ}}$.

The proof of the next Corollary follows directly from the work we did in the connected case in Section 6.2 and the methods used in the graded case generalizations of Section 4.4.

Corollary 6.14. *The functor* $D^R : \mathcal{HR}_m^o \to \mathcal{DR}_m^o$ *has a right adjoint* $U^R : \mathcal{DR}_m^o \to \mathcal{HR}_m^o$ *and the pair* (D^R, U^R) *forms an equivalence of categories.*

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Vita

The author is a native of Oporto, Portugal. He holds a Bachelor of Arts in Mathematics from the Instituto Superior Técnico, in Lisbon, Portugal, and a Master of Arts, also in Mathematics, from The State University of New York at Buffalo.