The Homology of the Spectrum *bo* and its Connective Covers

by

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Abstract

In this work we recompute the known ordinary mod 2 homology of the spectrum bo as a Hopf ring. In addition, we do the same for the connective covers $bo\langle 1\rangle$, $bo\langle 2\rangle$, and $bo\langle 4\rangle$. We compute this result using the Bar Spectral Sequence and explicit relations for the $\circ$-product and $\star$-product of our elements. We also use maps from our spectrum to the Eilenberg-Maclane spectrum $K(\mathbf{Z}/2)$ to simplify the result.

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Contents

0 Introduction

The object of this paper is to compute the Hopf ring $H_*\underline{bo}_*$ for the connective real Bott spectrum bo. There is an obvious map of spectra to the periodic Bott spectrum, $bo \to KO$. There is also the fundamental class

$$\Theta : bo \to K(\mathbf{Z}) \to H = K(\mathbf{Z}/2).$$

We show that the sum of these maps, $\Phi : bo \to KO \times H$, induces a *monomorphism* for all n,

$$\Phi_* : H_*\underline{bo}_n \to H_*\underline{KO}_n \otimes H_*\underline{H}_n,$$

and we compute $H_*\underline{bo}_n$ as a sub-Hopf ring of the known Hopf ring on the right, where we have used the notation $\underline{H}_n = K(\mathbf{Z}/2, n)$ and $H_*X = H_*(X; \mathbf{Z}/2)$. We find that unlike most Hopf rings that have been computed, $H_*\underline{bo}_*$ requires a large number of generators.

We start with some basic definitions and then proceed to the actual computation.

1 Definitions and Background Materials

1.1 Hopf rings

The following is a collection of basic facts about *Hopf rings*, from [1]. Let R be a graded associative commutative ring with unit. We let CoAlg_R be the category of graded cocommutative coassociative coalgebras with counit over R.

Let $H(*) = \{H_*(n)\}_{n\in \mathbf{Z}}$ be a Hopf ring over the ring R. Let $a \in H_i(n)$, $b \in H_j(k)$, and $c \in H_q(k)$.

Define $\deg x$ by $x \in H_{\deg x}(m)$. We sometimes write $\deg x$ as $|x|$.

(a) Each $H_*(n)$ is an element of CoAlg_R:

(i) There is a coassociative cocommutative coproduct for all n,

$$\Psi : H_*(n) \to H_*(n) \otimes H_*(n),$$

which we write as

$$\Psi(a) = \Sigma a' \otimes a''.$$

(ii) There is a counit, $\varepsilon : H_*(n) \to R$ such that $\Psi \circ (1_{H_*(n)} \otimes \varepsilon)$ is the identity.

(b) Each $H_*(k)$ is an abelian group object of CoAlg_R:

(i) There is a product

$$* : H_*(k) \otimes H_*(k) \to H_*(k)$$

which is associative and commutative.

(ii) The map $*$ is in CoAlg_R:

$$\Psi(b * c) = \Psi(b) * \Psi(c) = \Sigma(b' \otimes b'') * (c' \otimes c''),$$

where we use the usual $*$ product on $H_*(k) \otimes H_*(k)$, and

$$\varepsilon(b * c) = \varepsilon(b)\varepsilon(c).$$

(iii) The abelian group object unit, or zero, is $\eta : R \to H_*(k)$, which is in CoAlg_R. If we define $[0_k] = \eta(1) \neq 0$, then

$$[0_k] * b = b.$$

(iv) The conjugation $\chi : H_*(k) \to H_*(k)$ has $\chi\chi =$ identity and $\eta\varepsilon(b) = \Sigma b' * \chi(b'')$. It is the abelian group object inverse.

(c) There are associative maps

$$\circ : H_*(n) \otimes H_*(k) \to H_*(n + k)$$

with the properties:

(i) The map $\circ$ is in CoAlg_R:

$$\Psi(a \circ b) = \Psi(a) \circ \Psi(b) = \Sigma(a' \otimes a'') \circ (b' \otimes b'')$$

and

$$\varepsilon(a \circ b) = \varepsilon(a)\varepsilon(b).$$

(ii) Multiplication by zero: gives

$$[0_n] \circ b = \eta\varepsilon(b).$$

(iii) There is a unit map $e : R \to H_*(0)$. Define

$$e(1) = [1] \in H_0(0).$$

Then $[1] \circ b = b$.

(iv) Define $\chi([1]) = [-1] \in H_0(0)$. Then

$$\chi(a) = [-1] \circ a$$

and

$$\chi(a \circ b) = \chi(a) \circ b = a \circ \chi(b).$$

(v) Commutativity:

$$a \circ b = (-1)^{ij}[-1]^{\circ nk} \circ b \circ a = (-1)^{ij}\chi^{nk}(b \circ a),$$

where $a \in H_i(n)$ and $b \in H_j(k)$.

(vi) Distributivity:

$$a \circ (b * c) = \Sigma(-1)^{\deg a'' \deg b}(a' \circ b) * (a'' \circ c).$$

In our notation we use a^2 instead of a^{*2} and bc instead of $b * c$.

1.2 Ω-spectra

An Ω-*spectrum* E consists of a collection of H-spaces $\underline{E}_n$ such that for each $n \geq 0$ there is an isomorphism of H-spaces $\underline{E}_n \simeq \Omega \underline{E}_{n+1}$. Thus the spectrum bo has its spaces related by $\underline{bo}_n \simeq \Omega \underline{bo}_{n+1}$.

1.3 The elements $[x]$

Let C^0 be a homotopy category of topological spaces (with certain properties). Let $E_*(-)$ be an associative commutative multiplicative unreduced generalized homology theory with unit, and let $G^*(-)$ be a similar cohomology theory, both defined on C^0. Let E_* and G^* denote the two coefficient rings. Let $G^*(-)$ have a representing Ω-spectrum

$$\underline{G}_* = \{\underline{G}_n\}_{n\in\mathbf{Z}} \in GC^0,$$

i.e. $G^n(X) \simeq [X, \underline{G}_n]$ and $\Omega\underline{G}_{n+1} \simeq \underline{G}_n$ (with GC^0 the category of graded objects of C^0). Let $x \in G^n$ have degree $-n$ in the coefficient ring. Then $x \in G^n \simeq [\text{point}, \underline{G}_n]$ and so we have a map in homology $x_* : E_* \to E_*\underline{G}_n$. We define $[x] \in E_0\underline{G}_n$ to be the image of $1 \in E_*$ under this map.

If we let $z \in G^n$ and $x, y \in G^k$, then in [1] we have the results:

(i) $[z] \circ [x] = [zx] = [-1]^{\circ nk} \circ [x] \circ [z]$.

(ii) $[x] * [y] = [x+y] = [y+x] = [y] * [x]$.

(iii) $\Psi[z] = [z] \otimes [z]$.

(iv) The sub-Hopf algebra of $E_*\underline{G}_n$ generated by all $[x]$ with $x \in G^n$ is the group ring of G^n over E_*, i.e. $E_*[G^n]$ (using (ii)).

(v) The sub-Hopf algebra of $E_*\underline{G}_*$ generated by all $[x]$ with $x \in G^*$ is the ring ring of G^* over E_*, i.e. $E_*[G^*]$ (using (i) and (ii)).

1.4 Definition of the homotopy elements $[\alpha], [\beta]$ and $[\lambda]$

The homotopy elements α, β, and λ are defined by the relation:

$$KO_* = \mathbf{Z}[\alpha, \beta, \lambda^{\pm 1}]/(\alpha^3, 2\alpha, \alpha\beta, \beta^2 - 4\lambda),$$

$$\text{where } bo_* = \mathbf{Z}[\alpha, \beta, \lambda]/(\alpha^3, 2\alpha, \alpha\beta, \beta^2 - 4\lambda) \subset KO_*$$

and

$$|\alpha| = 1, |\beta| = 4, |\lambda| = 8.$$

These define the elements $[\alpha], [\beta]$, and $[\lambda]$. (We have used the notation $X_* = \pi_* X$.)

1.5 The elements $z_i, \overline{z}_i$ and e

We take $\mathbf{R}P^\infty = 1 \times BO(1) \subset 1 \times BO \subset \mathbf{Z} \times BO = \underline{bo}_0$. The elements $z_i \in H_i(\underline{bo}_0)$ are defined by:

$$H_* \mathbf{R}P^\infty = \mathbf{Z}/2\{z_i : i \geq 0\},$$

with $z_0 = [1] \in H_*(\mathbf{Z} \times BO)$. We also need the elements

$$\overline{z}_i = z_i * [-1].$$

Equivalently, $z_i = \overline{z}_i * [1]$. Thus $z_i \circ [v] = (\overline{z}_i \circ [v]) * [v]$ for any homotopy element v.

The fundamental class in $H_1 \underline{KO}_1$ is denoted e - this is also the suspension class. The element 1_1 is the unit for the star product in $H_0 \underline{KO}_1$.

1.6 Relationships of elements in $H_* \underline{KO}_*$

In [3] we learn of the following relationships:

(i) $1_1 = [0_1]$

(ii) $e^2 = e \circ z_1$

(iii) $(e^{\circ 2})^2 = e^{\circ 2} \circ z_2$

(iv) $(e^{\circ 3})^2 = 0$

(v) $e \circ [\alpha] = \overline{z}_1$

(vi) $e^{\circ 2} \circ [\beta] = \overline{z}_2 \circ [\alpha^2]$

(vii) $e^{\circ 4} \circ [\lambda] = \overline{z}_4 \circ [\beta]$

(viii) $\Psi(e) = e \otimes 1 + 1 \otimes e$

(ix) $\Psi z_k = \sum_{k=l+m} z_l \otimes z_m$

(x) $z_k \circ z_h = \binom{k+h}{k} z_{k+h} = \frac{(k+h)!}{k!h!} z_{k+h} \bmod 2$

(xi) $\overline{z}_1 \circ \overline{z}_1 = \overline{z}_1^2$

(xii) $z_1 \circ [\beta] = 0$

(xiii) $z_2 \circ [\beta] = 0$

(xiv) $z_1 \circ [\alpha^2] = 0.$

We also have the equality

$$(e^{\circ n})^2 = e^{\circ n} \circ z_n.$$

This only gives us a nonzero result if z_n is indecomposable, i.e. if $n = 2^i$.

We note that $e \circ z_i = e \circ \overline{z}_i$ for $i > 0$, since

$$e \circ \overline{z}_i = e \circ (z_i * [-1]) = (e \circ z_i) * (1 \circ [-1]) + (1 \circ z_i) * (e \circ [-1]) = e \circ z_i.$$

Also,

$$e \circ (\text{decomposable elements}) = 0,$$

where we call an element x decomposable if $x = ab$, with $\varepsilon a = \varepsilon b = 0$.

1.7 Types of algebras

In this paper $E(x_1, \ldots)$ is the exterior algebra on generators $(x_1, \ldots)$, and $P(x_1, \ldots)$ is the polynomial algebra on generators $(x_1, \ldots)$.

$\Gamma(y)$ is the divided power algebra, defined as follows: it has $\mathbf{Z}/2$-module generators $\gamma_i(y)$ for $i = 0, 1, 2, \ldots$, where $|y| = n$ and

(i) $\gamma_0(y) = 1$

(ii) $\gamma_1(y) = y$

(iii) $|\gamma_k(y)| = kn$

(iv) $\gamma_k(y)\gamma_h(y) = \binom{k+h}{k} \gamma_{k+h}(y)$.

We note that as an algebra, $\Gamma(y) = E\Big(\gamma_{2^j}(y) : j \geq 0\Big)$. (See [2].)

1.8 The Hopf ring for KO

In [3] we learn that space by space $H_*\underline{KO}_*$ has the following description:

$$H_*\underline{KO}_0 = H_*(\mathbf{Z} \times BO) = P\Big(z_k : k > 0\Big) \otimes P\Big([1], [1]^{-1}\Big)$$
$$H_*\underline{KO}_1 = H_*(U/O) = P\Big(e \circ z_{2k}\Big)$$
$$H_*\underline{KO}_2 = H_*(Sp/U) = P\Big(e^{\circ 2} \circ z_{4k}\Big)$$
$$H_*\underline{KO}_3 = H_*(Sp) = E\Big(e^{\circ 3} \circ z_{4k}\Big)$$
$$H_*\underline{KO}_4 = H_*(\mathbf{Z} \times BSp) = P\Big(z_{4k} \circ [\beta\lambda^{-1}]\Big) \otimes P\Big([\beta\lambda^{-1}], [\beta\lambda^{-1}]^{-1}\Big)$$
$$H_*\underline{KO}_5 = H_*(U/Sp) = E\Big(e \circ z_{4k} \circ [\beta\lambda^{-1}]\Big)$$
$$H_*\underline{KO}_6 = H_*(O/U) = E\Big(z_{2k} \circ [\alpha^2\lambda^{-1}], [\alpha^2\lambda^{-1}] - 1\Big)$$
$$H_*\underline{KO}_7 = H_*(O) = E\Big(z_k \circ [\alpha\lambda^{-1}], [\alpha\lambda^{-1}] - 1\Big)$$
$$H_*\underline{KO}_8 = H_*(\mathbf{Z} \times BO) = P\Big(z_k \circ [\lambda^{-1}] : k > 0\Big) \otimes P\Big([\lambda^{-1}], [\lambda^{-1}]^{-1}\Big).$$

1.9 The Frobenius and Verschiebung maps

If we are given A, a bicommutative Hopf algebra over $\mathbf{Z}/p$ (p prime), the *Frobenius map* $F : A \to A$ is defined by $F(x) = x^p$. Let A^* be the dual of A, $A^* = \mathrm{Hom}_{\mathbf{Z}/p}(A, \mathbf{Z}/p)$. We define the *Verschiebung map* $V : A \to A$ as the dual of F on A^*.

From [2] these maps have the following properties for $p = 2$:

(i) With a shift of grading V and F are Hopf algebra maps.

(ii) $VF = FV$:

$$V(x^{*2}) = VF(x) = FV(x) = [V(x)]^{*2}.$$

(iii) For the coalgebra $\Gamma(x)$,

$$V\Big(\gamma_{2q}(x)\Big) = \gamma_q(x)$$

and

$$V\Big(\gamma_q(x)\Big) = 0 \text{ if } q \neq 0 \bmod 2.$$

(iv) $F(x \circ V(y)) = F(x) \circ y$.

(v) $V(x \circ y) = V(x) \circ V(y)$.

1.10 The bigraded algebra $\mathrm{Tor}^R_{*,*}(\mathbf{Z}/2, \mathbf{Z}/2)$ and the suspension of x

When R is an augmented $\mathbf{Z}/2$-algebra we have the following properties:

(i) $\mathrm{Tor}^{P(x)}(\mathbf{Z}/2, \mathbf{Z}/2) = E[\sigma(x)]$.

(ii) $\mathrm{Tor}^{E(x)}(\mathbf{Z}/2, \mathbf{Z}/2) = \Gamma[\sigma(x)]$.

Generally, $\gamma_i(\sigma(y)) \in \mathrm{Tor}_{i,*}$ is defined whenever $y^2 = 0$.

(iii) $\mathrm{Tor}^{\mathbf{Z}/2[\mathbf{Z}]}(\mathbf{Z}/2,\mathbf{Z}/2) = E[\sigma(x)]$, where $\mathbf{Z}/2[\mathbf{Z}] = \mathbf{Z}/2[x, x^{-1}]$ is the group ring.

(iv) $\mathrm{Tor}^A \otimes \mathrm{Tor}^B \xrightarrow{\simeq} \mathrm{Tor}^{A\otimes B}$.

We keep in mind the *suspension* of $x, \sigma(x) \in \mathrm{Tor}^R_{1,*}(\mathbf{Z}/2,\mathbf{Z}/2)$, for any $x \in R$ such that $\varepsilon x = 0$. Then $\sigma(xy) = 0$ if $\varepsilon x = 0$ and $\varepsilon y = 0$, and $\sigma(z+xy) = \sigma(z)$. Also, $\sigma(x+y) = \sigma(x) + \sigma(y)$.

1.11 The bar spectral sequence

The *bar spectral sequence* is frequently used when we are dealing with a spectrum $\underline{E}_n$ and a field $\mathbf{F}$. It is a spectral sequence of Hopf algebras, with the basic property that:

$$E^2_{s,t} = \mathrm{Tor}^{H_*(\underline{E}_n;\mathbf{F})}_{s,t}(\mathbf{F},\mathbf{F}) \Rightarrow H_{s+t}(\underline{E}_{n+1};\mathbf{F}),$$

provided $\underline{E}_{n+1}$ is connected, with differentials

$$d_r : E^r_{s,t} \to E^r_{s-r,t+r-1}$$

For any $x \in H_*(\underline{E}_n)$, $e \circ x \in H_*(\underline{E}_{n+1})$ detects (the image in E^∞ of) $\sigma(x)$.

1.12 Maps between $H_*\underline{bo}_*$ and $H_*\underline{H}_*$

We have maps

$$\Theta : H_*\underline{bo}_k \to H_*K(\mathbf{Z},k) \to H_*\underline{H}_k,$$

which will be noted in each step as needed for future use.

The algebra generators of $H_*\underline{H}_*$ can be written uniquely in the form $\beta^{\circ j_0}_{(0)} \circ \beta^{\circ j_1}_{(1)} \circ \beta^{\circ j_2}_{(2)} \circ \ldots$, where $|\beta_{(i)}| = 2^i$. We order them lexicographically, according to the sequence of indices $(j_0, j_1, j_2, \ldots)$. Then whenever we introduce a new generator of the Hopf ring $H_*\underline{bo}_*$, we (generally) evaluate Θ on it, ignoring decomposables and retaining only the earliest indecomposable term in the ordering above. Thus

$$H_*\underline{H}_k = E(\beta_{(i_1)} \circ \ldots \circ \beta_{(i_k)} : i_k \geq \ldots \geq i_2 \geq i_1 \geq 0).$$

Also, $H_*\underline{H}_0$ has a basis consisting of 1 and [1], with $[1]^2 = 1$, and $H_*K(\mathbf{Z}, 0) = \mathbf{Z}[z_0, {z_0}^{-1}]$, where $z_0 = [1]$ and ${z_0}^{-1} = [-1]$. We note that Θ is a morphism of Hopf rings — it preserves $*$-products, $\circ$-products, etc. and it induces morphisms of bar spectral sequences.

We note that the suspension element of $H_*\underline{H}_*$ is $\beta_{(0)}$.

1.13 The notation $\alpha(i)$

The expression $\alpha(i) = k$ means that $i = 2^{i_1} + 2^{i_2} + \ldots + 2^{i_k}$, where $i_1 < i_2 < \ldots < i_k$. It counts the number of 1's in the binary expansion of i. Its significance is that $z_i = z_{2^{i_1}} \circ z_{2^{i_2}} \circ \ldots \circ z_{2^{i_k}}$, a k-fold $\circ$-product. We also use the notation $\alpha(i) = 0$ to mean $i = 0$.

2 The Computation of $H_*\underline{bo}_*$

We now start the computation.

STEP 0. We start with the fact that $\mathbf{Z}\times BO = \underline{bo}_0$. By [3], $H_*(\mathbf{Z}\times BO) = P\Big(z_i, {z_0}^{-1} : i \geq 0\Big)$. Thus

$$H_*\underline{bo}_0 = P\Big(z_i, {z_0}^{-1} : i \geq 0\Big).$$

Here the map

$$\Theta : H_*\underline{bo}_0 \to H_*K(\mathbf{Z}, 0) \to H_*\underline{H}_0$$

is clearly given by

$$\Theta(z_0) = [1], \Theta(z_0^{-1}) = [-1] = [1], \text{ and } \Theta(z_i) = 0$$

for $i > 0$.

2.1 The first cycle

In steps 1-8, we show that multiplication by $[\lambda]$,

$$\circ[\lambda] : H_*\underline{bo}_n \to H_*\underline{bo}_{n-8} = H_*\underline{KO}_{n-8} \simeq H_*\underline{KO}_n$$

is a monomorphism.

In steps 1, 2, 4 and 8, we find it useful to search for the lowest 0 digit in the binary expansion of i, by writing

$$i = 1 + 2 + \ldots + 2^{m-1} + 2^{m+1}q = 2^m(2q+1) - 1,$$

where $m \geq 0$ and $q \geq 0$.

STEP 1. Now we use the bar spectral sequence to find $H_*\underline{bo}_1$. Since

$$H_*\underline{bo}_0 = P\Big(z_i, {z_0}^{-1} : i \geq 0\Big) = P\Big(z_i : i \geq 1\Big) \otimes P\Big(z_0, {z_0}^{-1}\Big),$$

we have

$$E^2_{s,t} = \mathrm{Tor}^{H_*\underline{bo}_0}_{s,t}(\mathbf{Z}/2,\mathbf{Z}/2) = E\Big(\sigma(z_i) : i \geq 0\Big) \Rightarrow H_*\underline{bo}_1.$$

The suspension $\sigma(x)$ lies in the first filtration of the bar spectral sequence. Therefore the generators of the E^2-term are all in $E^2_{1,*}$, collapsing the bar spectral sequence. This gives $E\Big(\sigma(z_i) : i \geq 0\Big)$ in the E^∞-term.

The element $e \circ z_i \in H_*\underline{bo}_1$ detects $\sigma(z_i)$. To determine the algebra structure of $H_*\underline{bo}_1$, we start from the fact that $e^2 = e \circ z_1$, and use $V(z_{2i}) = z_i$, together with the fact that $F\Big(V(x) \circ y\Big) = x \circ F(y)$. Thus

$$(e \circ z_i)^2 = F(e \circ z_i) = F(e) \circ z_{2i} = e^2 \circ z_{2i} = (e \circ z_1) \circ z_{2i} = e \circ z_{2i+1}.$$

We apply this as often as possible to $e \circ z_i$ by writing $i = 2^m(2q+1) - 1$ as above; then $e \circ z_i = F^m(e \circ z_{2q})$.

Therefore

$$H_*\underline{bo}_1 = P\Big(e \circ z_{2i} : i \geq 0\Big).$$

For the new element e, we clearly have $\Theta(e) = \beta_{(0)}$.

STEP 2. To find $H_*\underline{bo}_2$: Since $H_*\underline{bo}_1 = P\Big(e \circ z_{2i} : i \geq 0\Big)$, the bar spectral sequence is

$$E^2_{s,t} = \mathrm{Tor}^{H_*\underline{bo}_1}_{s,t}(\mathbf{Z}/2,\mathbf{Z}/2) = E\Big(\sigma(e \circ z_{2i}) : i \geq 0\Big) \Rightarrow H_*\underline{bo}_2.$$

Once again the suspension $\sigma(x)$ lies in the first filtration, collapsing the bar spectral sequence at the E^2-term. This gives $E\Big(\sigma(e \circ z_{2i}) : i \geq 0\Big)$ in the E^∞-term.

The element $e^{\circ 2} \circ z_{2i} \in H_*\underline{bo}_2$ detects $\sigma(e \circ z_{2i})$. To determine the algebra structure of $H_*\underline{bo}_2$, we start from the fact that $(e^{\circ 2})^2 = e^{\circ 2} \circ z_2$, and use

$V(z_{4i}) = z_{2i}$. Thus

$$(e^{\circ 2} \circ z_{2i})^2 = F(e^{\circ 2} \circ z_{2i}) = F(e^{\circ 2}) \circ z_{4i} = (e^{\circ 2})^2 \circ z_{4i} = (e^{\circ 2} \circ z_2) \circ z_{4i} = e^{\circ 2} \circ z_{4i+2}.$$

We apply this as often as possible to $e^{\circ 2} \circ z_{2i}$ by writing $i = 2^m(2q+1) - 1$; then $e^{\circ 2} \circ z_{2i} = F^m(e^{\circ 2} \circ z_{4q})$.

We therefore have

$$H_* \underline{bo}_2 = P\Big(e^{\circ 2} \circ z_{4i} : i \geq 0\Big).$$

STEP 3. To find $H_* \underline{bo}_3$: Since $H_* \underline{bo}_2 = P\Big(e^{\circ 2} \circ z_{4i} : i \geq 0\Big)$, the bar spectral sequence is

$$E^2_{s,t} = \operatorname{Tor}^{H_* \underline{bo}_2}_{s,t}(\mathbf{Z}/2, \mathbf{Z}/2) = E\Big(\sigma(e^{\circ 2} \circ z_{4i}) : i \geq 0\Big) \Rightarrow H_* \underline{bo}_3.$$

The suspension $\sigma(e^{\circ 2} \circ z_{4i})$ collapses the bar spectral sequence at the E^2-term. Therefore, in the E^∞-term, we have $E\Big(\sigma(e^{\circ 2} \circ z_{4i}) : i \geq 0\Big)$.

As usual, $e^{\circ 3} \circ z_{4i} \in H_* \underline{bo}_3$ detects $\sigma(e^{\circ 2} \circ z_{4i})$. Since $(e^{\circ 3})^2 = 0$ in $H_* \underline{bo}_3$,

$$(e^{\circ 3} \circ z_{4i})^2 = F(e^{\circ 3} \circ z_{4i}) = F(e^{\circ 3}) \circ z_{8i} = (e^{\circ 3})^2 \circ z_{8i} = 0.$$

Thus

$$H_* \underline{bo}_3 = E\Big(e^{\circ 3} \circ z_{4i} : i \geq 0\Big).$$

STEP 4. To find $H_* \underline{bo}_4$: Since $H_* \underline{bo}_3 = E\Big(e^{\circ 3} \circ z_{4i} : i \geq 0\Big)$, the bar spectral sequence is

$$E^2_{s,t} = \operatorname{Tor}^{H_* \underline{bo}_3}_{s,t}(\mathbf{Z}/2, \mathbf{Z}/2) = \Gamma\Big(\sigma(e^{\circ 3} \circ z_{4i}) : i \geq 0\Big) \Rightarrow H_* \underline{bo}_4.$$

Once again the suspension $\sigma(x)$ lies in the first filtration, but the elements $\gamma_j\Big(\sigma(e^{\circ 3} \circ z_{4i})\Big)$ are in the j-th filtration. Since all of these elements are in *even* total degree $s+t$, the bar spectral sequence collapses at the E^2-term.

By the definition of a divided power algebra, the generators are the γ_{2^j}, so we have

$$E^\infty = \Gamma\Big(\sigma(e^{\circ 3} \circ z_{4i}) : i \geq 0\Big),$$

which can be rewritten as

$$E\Big(\gamma_{2^j}(\sigma(e^{\circ 3} \circ z_{4i})) : i, j \geq 0\Big).$$

To determine the algebra structure of $H_*\underline{bo}_4$, we start from the fact that $(e^{\circ 4})^2 = e^{\circ 4} \circ z_4$, and use $V(z_{8i}) = z_{4i}$. As usual, $e^{\circ 4} \circ z_{4i}$ detects $\sigma(e^{\circ 3} \circ z_{4i})$, giving

$$F(e^{\circ 4} \circ z_{4i}) = F(e^{\circ 4}) \circ z_{8i} = (e^{\circ 4} \circ z_4) \circ z_{8i} = e^{\circ 4} \circ z_{8i+4}.$$

We apply this as often as possible to $e^{\circ 4} \circ z_{4i}$ by writing $i = 2^m(2q+1) - 1$; then $e^{\circ 4} \circ z_{4i} = F^m(e^{\circ 4} \circ z_{8q})$.

Suppose $x \in H_*\underline{bo}_4$ detects $\gamma_{2^j}\Big(\sigma(e^{\circ 3} \circ z_{8q})\Big)$. Since

$$V^j\Big(\gamma_{2^j}(\sigma(e^{\circ 3} \circ z_{8q}))\Big) = \sigma(e^{\circ 3} \circ z_{8q}),$$

we must have $V^j x = e^{\circ 4} \circ z_{8q}$. Further,

$$V^j F^m x = F^m(e^{\circ 4} \circ z_{8q}) = e^{\circ 4} \circ z_{4i}$$

detects $\sigma(e^{\circ 3} \circ z_{4i})$ and $F^m x$ detects $\gamma_{2^j}\Big(\sigma(e^{\circ 3} \circ z_{4i})\Big)$. Thus the elements x (as j and q vary) are polynomial generators of $H_*\underline{bo}_4$.

To identify x, we consider the image of $H_*\underline{bo}_4$ under $\circ[\lambda]$ in the known algebra $H_*\underline{bo}_{-4} = H_*\underline{KO}_{-4} = P\Big(\overline{z}_{4i} \circ [\beta] : i > 0\Big) \otimes P\Big([\beta], [\beta]^{-1}\Big)$. We make use of the Hopf ring properties that

(i) $e^{\circ 4} \circ [\lambda] = \overline{z}_4 \circ [\beta]$

(ii) $z_{2k+1} \circ [\beta] = z_{2k} \circ z_1 \circ [\beta] = 0$

(iii) $z_{4k+2} \circ [\beta] = z_{4k} \circ z_2 \circ [\beta] = 0.$

Then

$$V^j(x \circ [\lambda]) = e^{\circ 4} \circ z_{8q} \circ [\lambda] = \overline{z}_4 \circ z_{8q} \circ [\beta] = \overline{z}_{8q+4} \circ [\beta] + \text{decomposables.}$$

Since $V(\overline{z}_{2k}) = \overline{z}_k$ and $V(\overline{z}_{2k+1}) = 0$, we conclude that

$$x \circ [\lambda] = \overline{z}_{2^j(8q+4)} \circ [\beta] + \ldots$$

We write

$$x = \overline{z}_{2^j(8q+4)} \circ [\beta\lambda^{-1}] + \ldots \in H_*\underline{KO}_4,$$

so that $F^m\Big(\overline{z}_{2^j(8q+4)} \circ [\beta\lambda^{-1}]\Big) + \ldots$ detects $\gamma_{2^j}\Big(\sigma(e^{\circ 3} \circ z_{4i})\Big)$.

We know that $\overline{z}_{2k+1} \circ [\beta] = 0$ and $\overline{z}_{4k+2} \circ [\beta] = 0$. Since any positive integer divisible by 4 has the form $2^j(8q+4)$ uniquely, the image of $H_*\underline{bo}_4$ is the whole polynomial ring $P\Big(\overline{z}_{4i} \circ [\beta] : i > 0\Big)$.

Thus we write formally

$$H_*\underline{bo}_4 = P\Big(\overline{z}_{4i} \circ [\beta\lambda^{-1}] : \alpha(i) \geq 1\Big) \subset H_*\underline{KO}_4.$$

Note that $[\beta\lambda^{-1}]$ itself is not an element of $H_*\underline{bo}_4$.

We clearly have $\Theta(\overline{z}_4 \circ [\beta\lambda^{-1}]) = \Theta(e^{\circ 4}) = {\beta_{(0)}}^{\circ 4}$. Since $V^i(\overline{z}_{2^{i+2}}) = \overline{z}_4$ and $V^i(\beta_{(i)}) = \beta_{(0)}$, we deduce

$$\Theta(\overline{z}_{2^{i+2}} \circ [\beta\lambda^{-1}]) = \beta_{(i)}^{\circ 4} + \ldots$$

where the unstated terms involve $\beta_{(j)}$ with $j < i$.

STEP 5. To find $H_*\underline{bo}_5$: Since $H_*\underline{bo}_4 = P\Big(\overline{z}_{4i} \circ [\beta\lambda^{-1}] : \alpha(i) \geq 1\Big)$, the bar spectral sequence is

$$E^2_{s,t} = \mathrm{Tor}^{H_*\underline{bo}_4}_{s,t}(\mathbf{Z}/2, \mathbf{Z}/2) = E\Big(\sigma(\overline{z}_{4i} \circ [\beta\lambda^{-1}]) : \alpha(i) \geq 1\Big) \Rightarrow H_*\underline{bo}_5.$$

Once again the suspension $\sigma(x)$ lies in the first filtration, which collapses the bar spectral sequence at the E^2-term.

Then $e \circ z_{4i} \circ [\beta\lambda^{-1}] = e \circ \overline{z}_{4i} \circ [\beta\lambda^{-1}]$ detects $\sigma(\overline{z}_{4i} \circ [\beta\lambda^{-1}])$. We have

$$F(e \circ z_{4i} \circ [\beta]) = F(e) \circ z_{8i} \circ [\beta] = (e \circ z_1) \circ z_{8i} \circ [\beta] = 0,$$

since $z_1 \circ [\beta] = 0$.

Therefore

$$H_*\underline{bo}_5 = E\Big(e \circ z_{4i} \circ [\beta\lambda^{-1}] : \alpha(i) \geq 1\Big).$$

Note that there is an injection from $H_*\underline{bo}_5$ to

$$H_*\underline{bo}_{-3} = H_*\underline{KO}_{-3} = E\Big(e \circ z_{4k} \circ [\beta] : k \geq 0\Big).$$

We also have

$$\Theta(e \circ \overline{z}_{2^{i+2}} \circ [\beta\lambda^{-1}]) = \beta_{(0)} \circ \beta_{(i)}^{\circ 4} + \ldots$$

STEP 6. To find $H_*\underline{bo}_6$: Since $H_*\underline{bo}_5 = E\Big(e \circ z_{4i} \circ [\beta\lambda^{-1}] : \alpha(i) \geq 1\Big)$, the bar spectral sequence is

$$E^2_{s,t} = \mathrm{Tor}^{H_*\underline{bo}_5}_{s,t}(\mathbf{Z}/2, \mathbf{Z}/2) = \Gamma\Big(\sigma(e \circ z_{4i} \circ [\beta\lambda^{-1}]) : \alpha(i) \geq 1\Big) \Rightarrow H_*\underline{bo}_6.$$

The suspension $\sigma(x)$ lies in the first filtration, but the elements $\gamma_j(x)$ are in the j-th filtration. Since all elements have even total degree, the bar spectral sequence collapses at the E^2-term. Thus

$$E^\infty = \Gamma\Big(\sigma(e \circ z_{4i} \circ [\beta\lambda^{-1}]) : \alpha(i) \geq 1\Big),$$

which can be rewritten as

$$E\Big(\gamma_{2^j}(\sigma(e \circ z_{4i} \circ [\beta\lambda^{-1}])) : \alpha(i) \geq 1, j \geq 0\Big).$$

As usual, $e^{\circ 2} \circ \overline{z}_{4i} \circ [\beta\lambda^{-1}] = e^{\circ 2} \circ z_{4i} \circ [\beta\lambda^{-1}] \in H_*\underline{bo}_6$ detects $\sigma(e \circ z_{4i} \circ [\beta\lambda^{-1}])$. We use the Hopf ring fact that $e^{\circ 2} \circ [\beta] = \overline{z}_2 \circ [\alpha^2]$ to rewrite this as

$$\overline{z}_2 \circ \overline{z}_{4i} \circ [\alpha^2\lambda^{-1}] = \overline{z}_{4i+2} \circ [\alpha^2\lambda^{-1}] + \dots$$

Again we apply $\circ[\lambda]$ to map $H_*\underline{bo}_6$ into the known algebra

$$H_*\underline{bo}_{-2} = H_*\underline{KO}_{-2} = E\Big(\overline{z}_{2i} \circ [\alpha^2] : i > 0\Big) \otimes E\Big([\alpha^2] - 1\Big).$$

We know that $\overline{z}_{2i+1} \circ [\alpha^2] = 0$, so no odd $\overline{z}$'s appear. Suppose $x \in H_*\underline{bo}_6$ detects $\gamma_{2^j}\Big(\sigma(e \circ z_{4i} \circ [\beta\lambda^{-1}])\Big)$. Then

$$V^j x = e^{\circ 2} \circ \overline{z}_{4i} \circ [\beta\lambda^{-1}] = \overline{z}_{4i+2} \circ [\alpha^2\lambda^{-1}] + \dots$$

and

$$V^j(x \circ [\lambda]) = \overline{z}_{4i+2} \circ [\alpha^2] + \dots$$

Thus, we must have $x \circ [\lambda] = \overline{z}_{2^j(4i+2)} \circ [\alpha^2] + \dots$

Thus $H_*\underline{bo}_6 \to H_*\underline{bo}_{-2}$ is monic and we can identify $H_*\underline{bo}_6$ with its image in $H_*\underline{bo}_{-2}$. Every number $2k$ with $\alpha(k) \geq 2$ can be written uniquely in the form $2^j(4i+2)$ with $i > 0$. Hence

$$H_*\underline{bo}_6 = E\Big(\overline{z}_{2i} \circ [\alpha^2\lambda^{-1}] + \dots : \alpha(i) \geq 2\Big),$$

where $\overline{z}_{2^j(4i+2)} \circ [\alpha^2\lambda^{-1}] + \dots$ detects $\gamma_{2^j}\Big(\sigma(e \circ z_{4i} \circ [\beta\lambda^{-1}])\Big)$.

Clearly,

$$\Theta(\overline{z}_2 \circ \overline{z}_{2^{i+2}} \circ [\alpha^2\lambda^{-1}]) = \Theta(e^{\circ 2} \circ \overline{z}_{2^{i+2}} \circ [\beta\lambda^{-1}]) = \beta_{(0)}^{\circ 2} \circ \beta_{(i)}^{\circ 4} + \dots$$

Since Θ commutes with V^j and $\overline{z}_2 \circ \overline{z}_{2^{i+2}} = \overline{z}_{2+2^{i+2}} + \dots$, we deduce

$$\Theta\Big(\overline{z}_{2^{j+1}+2^{i+j+2}} \circ [\alpha^2\lambda^{-1}]\Big) = \beta_{(j)}^{\circ 2} \circ \beta_{(i+j)}^{\circ 4} + \dots$$

We rewrite this as

$$\Theta\Big(\overline{z}_{2^{i_1+1}+2^{i_2+2}} \circ [\alpha^2\lambda^{-1}]\Big) = \beta_{(i_1)}^{\circ 2} \circ \beta_{(i_2)}^{\circ 4} + \dots$$

for $i_2 \geq i_1 \geq 0$.

STEP 7. To find $H_*\underline{bo}_7$: Since $H_*\underline{bo}_6 = E\Big(\overline{z}_{2i} \circ [\alpha^2\lambda^{-1}] + \dots : \alpha(i) \geq 2\Big)$, the bar spectral sequence is

$$E^2_{s,t} = \mathrm{Tor}^{H_*\underline{bo}_6}_{s,t}(\mathbf{Z}/2, \mathbf{Z}/2) = \Gamma\Big(\sigma(\overline{z}_{2i} \circ [\alpha^2\lambda^{-1}]) : \alpha(i) \geq 2\Big) \Rightarrow H_*\underline{bo}_7.$$

We rewrite

$$\Gamma\Big(\sigma(\overline{z}_{2i} \circ [\alpha^2\lambda^{-1}]) : \alpha(i) \geq 2\Big)$$

as

$$E\Big(\gamma_{2^j}(\sigma(\overline{z}_{2i} \circ [\alpha^2\lambda^{-1}])) : \alpha(i) \geq 2, j \geq 0\Big).$$

The suspension $\sigma(x)$ lies in the first filtration, but the elements $\gamma_{2^j}(x)$ are in the 2^j-th filtration. Thus, for the moment, we gain no information about the behavior of the bar spectral sequence.

As usual, $e \circ \overline{z}_{2i} \circ [\alpha^2\lambda^{-1}]$ detects $\sigma(\overline{z}_{2i} \circ [\alpha^2\lambda^{-1}])$. We now use the Hopf ring fact that $e \circ [\alpha] = \overline{z}_1$ to write this as

$$\overline{z}_1 \circ \overline{z}_{2i} \circ [\alpha\lambda^{-1}] = \overline{z}_{2i+1} \circ [\alpha\lambda^{-1}] + \dots$$

Again we apply $\circ[\lambda]$ to map $H_*\underline{bo}_7$ into the known algebra

$$H_*\underline{bo}_{-1} = H_*\underline{KO}_{-1} = E\Big(\overline{z}_i \circ [\alpha] : i > 0\Big) \otimes E\Big([\alpha] - 1\Big).$$

Since $\underline{bo}_{-1}$ maps as a subset into a spectral sequence that collapses, the bar spectral sequence for $\underline{bo}_7$ collapses.

Suppose that $x \in H_*\underline{bo}_7$ detects $\gamma_{2^j}\Big(\sigma(\overline{z}_{2i} \circ [\alpha^2\lambda^{-1}])\Big)$. Then

$$V^j x = e \circ \overline{z}_{2i} \circ [\alpha^2\lambda^{-1}] = \overline{z}_1 \circ \overline{z}_{2i} \circ [\alpha\lambda^{-1}] = \overline{z}_{2i+1} \circ [\alpha\lambda^{-1}] + \dots$$

and

$$V^j(x \circ [\lambda]) = \overline{z}_{2i+1} \circ [\alpha] + \dots$$

We must have $x \circ [\lambda] = \overline{z}_{2^j(2i+1)} \circ [\alpha] + \dots$, where $\alpha(i) \geq 2$. Every number k with $\alpha(k) \geq 3$ can be written uniquely in the form $2^j(2i+1)$ with $\alpha(i) \geq 2$. Thus $H_*\underline{bo}_7 \to H_*\underline{bo}_{-1}$ is monic and we write

$$H_*\underline{bo}_7 = E\Big(\overline{z}_i \circ [\alpha\lambda^{-1}] + \dots : \alpha(i) \geq 3\Big).$$

Clearly,

$$\Theta\Big(\overline{z}_1 \circ \overline{z}_{2^{i_1+1}+2^{i_2+2}} \circ [\alpha\lambda^{-1}]\Big) = \Theta\Big(e \circ \overline{z}_{2^{i_1+1}+2^{i_2+2}} \circ [\alpha^2\lambda^{-1}]\Big) = \beta_{(0)} \circ \beta_{(i_1)}^{\circ 2} \circ \beta_{(i_2)}^{\circ 4} + \dots$$

for $i_2 \geq i_1$. Since Θ commutes with V^j, we deduce

$$\Theta\Big(\overline{z}_{2^j+2^{j+i_1+1}+2^{j+i_2+2}} \circ [\alpha\lambda^{-1}]\Big) = \beta_{(j)} \circ \beta_{(j+i_1)}^{\circ 2} \circ \beta_{(j+i_2)}^{\circ 4} + \dots$$

For any i with $\alpha(i) = 3$, we therefore have, after reindexing,

$$\Theta(\overline{z}_i \circ [\alpha\lambda^{-1}]) = \beta_{(i_1)} \circ \beta_{(i_2)}^{\circ 2} \circ \beta_{(i_3)}^{\circ 4} + \dots$$

where $i = 2^{i_1} + 2^{i_2+1} + 2^{i_3+2}$ is the binary expansion of i with $i_3 \geq i_2 \geq i_1$.

STEP 8. To find $H_*\underline{bo}_8$: Since $H_*\underline{bo}_7 = E\Big(\overline{z}_i \circ [\alpha\lambda^{-1}] + \dots : \alpha(i) \geq 3\Big)$, the bar spectral sequence is

$$E^2_{s,t} = \mathrm{Tor}^{H_*\underline{bo}_7}_{s,t}(\mathbf{Z}/2, \mathbf{Z}/2) = \Gamma\Big(\sigma(\overline{z}_i \circ [\alpha\lambda^{-1}]) : \alpha(i) \geq 3\Big) \Rightarrow H_*\underline{bo}_8.$$

We note that

$$\Gamma\Big(\sigma(\overline{z}_i \circ [\alpha\lambda^{-1}]) : \alpha(i) \geq 3\Big)$$

can be rewritten as

$$E\Big(\gamma_{2^j}(\sigma(\overline{z}_i \circ [\alpha\lambda^{-1}])) : \alpha(i) \geq 3, j \geq 0\Big).$$

As usual, $\sigma(\overline{z}_i \circ [\alpha\lambda^{-1}])$ is detected by $e \circ \overline{z}_i \circ [\alpha\lambda^{-1}]$. Since $e \circ [\alpha] = \overline{z}_1$, we can rewrite this as $\overline{z}_1 \circ \overline{z}_i \circ [\lambda^{-1}]$.

We wish to simplify $\overline{z}_1 \circ \overline{z}_i \circ [\lambda^{-1}]$ by using the fact that

$$\overline{z}_1 \circ \overline{z}_{2i} = \overline{z}_{2i+1} + \text{ decomposables.}$$

Then

$$\overline{z}_1 \circ \overline{z}_{2i+1} = \overline{z}_1 \circ \overline{z}_1 \circ \overline{z}_{2i} + \overline{z}_1 \circ \text{ decomposables.}$$

Since $\Psi(\overline{z}_1) = \overline{z}_1 \otimes 1 + 1 \otimes \overline{z}_1$, we have $\overline{z}_1 \circ$ decomposables $= 0$. Hence

$$\overline{z}_1 \circ \overline{z}_{2i+1} \circ [\lambda^{-1}] = \overline{z}_1 \circ \overline{z}_1 \circ \overline{z}_{2i} \circ [\lambda^{-1}] = F(\overline{z}_1) \circ \overline{z}_{2i} \circ [\lambda^{-1}] = F(\overline{z}_1 \circ \overline{z}_i \circ [\lambda^{-1}]).$$

We apply this relation as often as possible, by expanding i in binary form and looking for the lowest 0 digit: $i = 2^m(2q+1) - 1$, where $q, m \geq 0$. Then

$$\overline{z}_1 \circ \overline{z}_i \circ [\lambda^{-1}] = F^m(\overline{z}_1 \circ \overline{z}_{2q} \circ [\lambda^{-1}]) = F^m(\overline{z}_{2q+1} \circ [\lambda^{-1}]) + \ldots$$

Here $m + \alpha(2q+1) = m + \alpha(q) + 1 = \alpha(i) + 1 \geq 4$.

To see that the bar spectral sequence collapses, apply $\circ[\lambda]$ to map to the known bar spectral sequence for H_*BO, which does collapse. As before, $\gamma_{2^j}\Big(\sigma(\overline{z}_i \circ [\alpha\lambda^{-1}])\Big)$ is detected by

$$F^m\Big(e \circ \overline{z}_i \circ [\alpha\lambda^{-1}]\Big) + \ldots = F^m\Big(\overline{z}_{2^j(2q+1)} \circ [\lambda^{-1}]\Big) + \ldots$$

Therefore,

$$\begin{aligned} H_*\underline{bo}_8 =& P\Big(\overline{z}_i \circ [\lambda^{-1}] + \ldots : \alpha(i) \geq 4\Big) \\ & \otimes P\Big(F^j(\overline{z}_i \circ [\lambda^{-1}]) + \ldots : \alpha(i) + j = 4, i, j \geq 1\Big). \end{aligned}$$

We need the value of Θ on $F^m(\overline{z}_k \circ [\lambda^{-1}]) + \ldots$ whenever $\alpha(k) + m = 4$ (including the case $m = 0$). As above, we write $k = 2^j(2q+1)$, so that

$\alpha(k) = \alpha(q) + 1$. Then

$$V^j\Big(F^m(\overline{z}_k \circ [\lambda^{-1}])\Big) = F^m(\overline{z}_{2q+1} \circ [\lambda^{-1}]) = \overline{z}_1 \circ \overline{z}_i \circ [\lambda^{-1}] = e \circ \overline{z}_i \circ [\alpha\lambda^{-1}],$$

where

$$i = 2^m(2q+1) - 1 = (2^m - 1) + 2^{m+1}q,$$

so that $\alpha(i) = m + \alpha(q) = m + \alpha(k) - 1 = 3$. Thus we let

$$i = 2^{i_1} + 2^{i_2+1} + 2^{i_3+2}$$

be the binary expansion of i, where $i_3 \geq i_2 \geq i_1$. We know

$$\Theta(e \circ \overline{z}_i \circ [\alpha\lambda^{-1}]) = \beta_{(0)} \circ \beta_{(i_1)} \circ \beta_{(i_2)}^{\circ 2} \circ \beta_{(i_3)}^{\circ 4} + \ldots$$

Since Θ commutes with V, we must have

$$\Theta\Big(F^m(\overline{z}_k \circ [\lambda^{-1}])\Big) = \beta_{(j)} \circ \beta_{(j+i_1)} \circ \beta_{(j+i_2)}^{\circ 2} \circ \beta_{(j+i_3)}^{\circ 4} + \ldots$$

We reindex as

$$\Theta\Big(F^j(\overline{z}_i \circ [\lambda^{-1}])\Big) = \beta_{(i_1)} \circ \beta_{(i_2)} \circ \beta_{(i_3)}^{\circ 2} \circ \beta_{(i_4)}^{\circ 4} + \ldots$$

whenever $\alpha(i) + j = 4$. Here, we write $i = 2^{i_1}(2q+1)$, where 2^{i_1} is the largest power of 2 that divides i, and use the binary expansion

$$2^j i - 2^{i_1} = (2^{j+i_1} - 2^{i_1}) + 2^{j+i_1+1}q = 2^{i_2} + 2^{i_3+1} + 2^{i_4+2},$$

where $i_4 \geq i_3 \geq i_2 \geq i_1$. We note that $i_2 > i_1$ if $j = 0$, but that $i_2 = i_1$ if $j > 0$.

2.2 The second cycle

From step 9 on, $\circ[\lambda]$ will no longer be a monomorphism. Instead, we treat $H_*\underline{bo}_n$ as a sub-Hopf algebra of $H_*\underline{bo}_{n-8} \otimes H_*\underline{H}_n \subset H_*\underline{KO}_n \otimes H_*\underline{H}_n$ by means of $\circ[\lambda]$ and Θ. To check that the bar spectral sequence for $H_*\underline{bo}_n$ collapses, it is only necessary to verify that the map $\underline{bo}_{n-1} \to \underline{KO}_{n-1} \times \underline{H}_{n-1}$ induces a monomorphism on the E^2-terms, thus embedding the spectral sequence in one that is known to collapse. This will be clear in practice. We make frequent appeal to the structure of $H_*\underline{bo}_{n-8}$. Each factor of $H_*\underline{bo}_{n-1}$ will lead to one or more factors of $H_*\underline{bo}_n$.

STEP 9. To find $H_*\underline{bo}_9$: Since $H_*\underline{bo}_8$ is known, the bar spectral sequence is

$$\begin{aligned} E^2_{s,t} &= \operatorname{Tor}^{H_*\underline{bo}_8}_{s,t}(\mathbf{Z}/2, \mathbf{Z}/2) \\ &= E\Big(\sigma(\overline{z}_i \circ [\lambda^{-1}]) : \alpha(i) \geq 4\Big) \\ &\quad \otimes E\Big(\sigma(F^j(\overline{z}_i \circ [\lambda^{-1}])) : \alpha(i) + j = 4, i, j \geq 1\Big) \\ &\Rightarrow H_*\underline{bo}_9. \end{aligned}$$

This is the last step in which we will explicitly state the bar spectral sequence.

As usual, $e \circ z_i \circ [\lambda^{-1}] + \ldots = e \circ \overline{z}_i \circ [\lambda^{-1}] + \ldots$ detects $\sigma(\overline{z}_i \circ [\lambda^{-1}])$. We use the relation $e \circ z_{2i+1} = F(e \circ z_i)$ as often as possible, as in step 1. Write $i = 2^m(2q+1) - 1$ where $\alpha(i) = m + \alpha(q) \geq 4$; then

$$e \circ z_i \circ [\lambda^{-1}] + \ldots = F^m(e \circ z_{2q} \circ [\lambda^{-1}]) + \ldots$$

detects $\sigma(\overline{z}_i \circ [\lambda^{-1}])$. Thus if $\alpha(q) \geq 4$, the element $e \circ z_{2q} \circ [\lambda^{-1}] + \ldots$ is a polynomial generator of $H_*\underline{bo}_9$. But if $\alpha(q) < 4$ the polynomial generator is $F^m(e \circ z_{2q} \circ [\lambda^{-1}]) + \ldots$ instead, with $m = 4 - \alpha(q)$. This takes care of the first exterior algebra in $E^2 = E^\infty$.

The method of applying $\circ[\lambda]$ *fails* for the second exterior algebra. Although $F^j(\overline{z}_i \circ [\lambda^{-1}]) + \dots$ is a generator of $H_*\underline{bo}_8$, it becomes decomposable in $H_*\underline{bo}_0$ and $\circ[\lambda]$ annihilates $\sigma\Big(F^j(\overline{z}_i \circ [\lambda^{-1}])\Big)$ at the E^2-level. Instead, we apply the morphism Θ of bar spectral sequences to the bar spectral sequence for $H_*\underline{H}_9$. Suppose $x \in H_*\underline{bo}_9$ is the element that detects $\sigma\Big(F^j(\overline{z}_i \circ [\lambda^{-1}])\Big)$; then $x \circ [\lambda] = 0$, and by step 8, $\Theta(x)$ detects $\sigma(\beta_{(i_1)} \circ \beta_{(i_2)} \circ \beta_{(i_3)}^{\circ 2} \circ \beta_{(i_4)}^{\circ 4} + \dots)$ and therefore is

$$\Theta(x) = \beta_{(0)} \circ \beta_{(i_1)} \circ \beta_{(i_2)} \circ \beta_{(i_3)}^{\circ 2} \circ \beta_{(i_4)}^{\circ 4} + \dots$$

Also, since $j > 0$, we have $i_2 = i_1$ here. Hence $\circ[\lambda]$ and Θ define a monomorphism

$$H_*\underline{bo}_9 \to H_*\underline{KO}_9 \otimes H_*\underline{H}_9 = H_*(\underline{KO}_9 \times \underline{H}_9)$$

which makes it clear that $x^2 = 0$, as $H_*\underline{H}_9$ is an exterior algebra. We therefore treat $H_*\underline{bo}_9$ as a subalgebra of $H_*\underline{KO}_9 \otimes H_*\underline{H}_9$, and label those generators that map trivially to $H_*\underline{KO}_9$ by their images under Θ instead. Thus, we write

$$\begin{aligned} H_*\underline{bo}_9 =& P\Big(e \circ z_{2i} \circ [\lambda^{-1}] : \alpha(i) \geq 4\Big) \\ & \otimes P\Big(F^j(e \circ z_{2i} \circ [\lambda^{-1}]) + \dots : \alpha(i) + j = 4, j \geq 1\Big) \\ & \otimes E\Big(\beta_{(i_1)} \circ \beta_{(i_2)}^{\circ 2} \circ \beta_{(i_3)}^{\circ 2} \circ \beta_{(i_4)}^{\circ 4} + \dots : i_4 \geq i_3 \geq i_2 \geq i_1 = 0\Big). \end{aligned}$$

The generators of the third factor are already defined by their images under Θ. We need the images of the generators of the form $F^j(e \circ z_{2i} \circ [\lambda^{-1}]) + \dots$ with $\alpha(i) + j = 4$ (including the case $j = 0$). We again use $F^j(e \circ z_{2i}) = e \circ z_k$, where $k = 2^j(2i+1) - 1$, and note that $\alpha(k) = j + \alpha(i) = 4$. By step 8,

$$\begin{aligned} \Theta\Big(F^j(e \circ z_{2i} \circ [\lambda^{-1}]) + \dots\Big) &= \Theta(e \circ z_k \circ [\lambda^{-1}] + \dots) \\ &= \beta_{(i_1)} \circ \beta_{(i_2)} \circ \beta_{(i_3)} \circ \beta_{(i_4)}^{\circ 2} \circ \beta_{(i_5)}^{\circ 4} + \dots \end{aligned}$$

with indices defined by the binary expansion $k = 2^{i_2} + 2^{i_3} + 2^{i_4+1} + 2^{i_5+2}$, with $i_5 \geq i_4 \geq i_3 > i_2 \geq i_1 = 0$. Since $i_3 > i_2$, this is different from all the generators in the third factor of $H_*\underline{bo}_9$.

STEP 10. To find $H_*\underline{bo}_{10}$: We use $H_*\underline{bo}_9$.

The first factor in E^2 is

$$E\Big(\sigma(e \circ z_{2i} \circ [\lambda^{-1}]) : \alpha(i) \geq 4\Big),$$

where $e^{\circ 2} \circ z_{2i} \circ [\lambda^{-1}]$ detects $\sigma(e \circ z_{2i} \circ [\lambda^{-1}])$. As in step 2 we write $i = 2^m(2q+1) - 1$ where $\alpha(i) = m + \alpha(q) \geq 4$; then

$$e^{\circ 2} \circ z_{2i} \circ [\lambda^{-1}] = F^m(e^{\circ 2} \circ z_{4q} \circ [\lambda^{-1}])$$

detects $\sigma(e \circ z_{2i} \circ [\lambda^{-1}])$. As in step 9 (with indices doubled), we deduce polynomial generators $e^{\circ 2} \circ z_{4q} \circ [\lambda^{-1}]$ (with $\alpha(q) \geq 4$) and $F^m(e^{\circ 2} \circ z_{4q} \circ [\lambda^{-1}])$ (with $\alpha(q) + m = 4, m \geq 1$) in $H_*\underline{bo}_{10}$.

The second factor in E^2 is

$$E\Big(\sigma(F^j(e \circ z_{2i} \circ [\lambda^{-1}])) : \alpha(i) + j = 4, j \geq 1\Big),$$

which is annihilated by $\circ[\lambda]$, so we therefore apply Θ instead. By step 9,

$$\begin{aligned}\Theta\Big(\sigma(e \circ z_k \circ [\lambda^{-1}]) + \ldots\Big) &= \Theta\Big(\sigma(F^j(e \circ z_{2i} \circ [\lambda^{-1}]))\Big) \\ &= \sigma(\beta_{(0)} \circ \beta_{(i_2)} \circ \beta_{(i_3)} \circ \beta_{(i_4)}^{\circ 2} \circ \beta_{(i_5)}^{\circ 4} + \ldots)\end{aligned}$$

where we use the binary expansion

$$k = 2^j(2i+1) - 1 = 2^{i_2} + 2^{i_3} + 2^{i_4+1} + 2^{i_5+2}$$

with $i_5 \geq i_4 \geq i_3 > i_2$. Since $j > 0$, k is odd and $i_2 = 0$. This element is therefore detected by $\beta_{(0)}^{\circ 3} \circ \beta_{(i_3)} \circ \beta_{(i_4)}^{\circ 2} \circ \beta_{(i_5)}^{\circ 4} + \ldots$

We rewrite the third factor

$$\Gamma\Big(\sigma(\beta_{(i_1)} \circ \beta_{(i_2)}^{\circ 2} \circ \beta_{(i_3)}^{\circ 2} \circ \beta_{(i_4)}^{\circ 4}) : i_4 \geq i_3 \geq i_2 \geq i_1 = 0\Big)$$

in E^2 as

$$E\Big(\gamma_{2^j}(\sigma(\beta_{(0)}\circ\beta_{(i_2)}^{\circ 2}\circ\beta_{(i_3)}^{\circ 2}\circ\beta_{(i_4)}^{\circ 4})) : i_4 \geq i_3 \geq i_2 \geq 0, j \geq 0\Big).$$

The generator shown is thus detected by

$$\beta_{(j)}\circ\beta_{(j)}\circ\beta_{(j+i_2)}^{\circ 2}\circ\beta_{(j+i_3)}^{\circ 2}\circ\beta_{(j+i_4)}^{\circ 4} \ + \ \ldots$$

Generally, a factor

$$E(\beta_{(j_1)}\circ\ldots\circ\beta_{(j_m)})$$

in $H_*\underline{bo}_n$ gives a factor

$$\Gamma\Big(\sigma(\beta_{(j_1)}\circ\ldots\circ\beta_{(j_m)})\Big)$$

in E^2. This yields the factor

$$E\Big(\beta_{(j)}\circ\beta_{(j+j_1)}\circ\ldots\circ\beta_{(j+j_m)} : j \geq 0\Big)$$

in $H_*\underline{bo}_{n+1}$, which we reindex. In future, we use this without comment.

Therefore,

$$\begin{aligned} H_*\underline{bo}_{10} =& P\Big(e^{\circ 2}\circ z_{4i}\circ[\lambda^{-1}] : \alpha(i)\geq 4\Big) \\ &\otimes P\Big(F^j(e^{\circ 2}\circ z_{4i}\circ[\lambda^{-1}]) + \ldots : \alpha(i)+j = 4, j\geq 1\Big) \\ &\otimes E\Big(\beta_{(i_1)}^{\circ 3}\circ\beta_{(i_2)}\circ\beta_{(i_3)}^{\circ 2}\circ\beta_{(i_4)}^{\circ 4} : i_4\geq i_3\geq i_2 > i_1 = 0\Big) \\ &\otimes E\Big(\beta_{(i_1)}^{\circ 2}\circ\beta_{(i_2)}^{\circ 2}\circ\beta_{(i_3)}^{\circ 2}\circ\beta_{(i_4)}^{\circ 4} : i_4\geq i_3\geq i_2\geq i_1\geq 0\Big), \end{aligned}$$

and we again have a monomorphism

$$H_*\underline{bo}_{10} \to H_*\underline{KO}_2 \otimes H_*\underline{H}_{10} = H_*(\underline{KO}_{10}\times\underline{H}_{10})$$

We need to compute $\Theta\Big(F^j(e^{\circ 2}\circ z_{4i}\circ[\lambda^{-1}])\Big)$ whenever $\alpha(i)+j = 4$ and $i, j\geq 0$. Since

$$F^j(e^{\circ 2}\circ z_{4i}\circ[\lambda^{-1}]) = e^{\circ 2}\circ z_{2k}\circ[\lambda^{-1}] = e\circ(e\circ z_{2k}\circ[\lambda^{-1}]),$$

where $k = 2^j(2i+1)-1$ (thus giving $\alpha(k) = j + \alpha(i) = 4$), step 9 gives

$$\Theta\Big(F^j(e^{\circ 2} \circ z_{4i} \circ [\lambda^{-1}])\Big) = \beta_{(i_1)}^{\circ 2} \circ \beta_{(i_2)} \circ \beta_{(i_3)} \circ \beta_{(i_4)}^{\circ 2} \circ \beta_{(i_5)}^{\circ 4} + \ldots,$$

where we use the binary expansion $2k = 2^{i_2} + 2^{i_3} + 2^{i_4+1} + 2^{i_5+2}$ with indices $i_5 \geq i_4 \geq i_3 > i_2 > i_1 = 0$. Again, we observe that the conditions on the indices ensure that this element differs from all the previously mentioned generators of $H_*\underline{bo}_{10}$. (In the future, we suppress any mention of this point.)

STEP 11. To find $H_*\underline{bo}_{11}$: We use $H_*\underline{bo}_{10}$.

The factor $P\Big(e^{\circ 2} \circ z_{4i} \circ [\lambda^{-1}] : \alpha(i) \geq 4\Big)$ in $H_*\underline{bo}_{10}$ leads to

$$E\Big(e^{\circ 3} \circ z_{4i} \circ [\lambda^{-1}] : \alpha(i) \geq 4\Big),$$

just as in step 3.

The factor

$$P\Big(F^j(e^{\circ 2} \circ z_{4i} \circ [\lambda^{-1}]) + \ldots : \alpha(i) + j = 4, j \geq 1\Big)$$

gives rise to the factor

$$E\Big(\sigma(F^j(e^{\circ 2} \circ z_{4i} \circ [\lambda^{-1}])) : \alpha(i) + j = 4, j \geq 1\Big)$$

in E^2. However, $F^j(e^{\circ 2} \circ z_{4i} \circ [\lambda^{-1}])$ decomposes in $H_*\underline{bo}_2$ and we must apply Θ instead. By step 10, $\sigma\Big(F^j(e^{\circ 2} \circ z_{4i} \circ [\lambda^{-1}])\Big)$ is detected by

$$\beta_{(0)} \circ (\beta_{(0)}^{\circ 2} \circ \beta_{(i_2)} \circ \beta_{(i_3)} \circ \beta_{(i_4)}^{\circ 2} \circ \beta_{(i_5)}^{\circ 4}),$$

where $i_5 \geq i_4 \geq i_3 > i_2$. Since $j > 0$, we have $i_2 = 1$.

The two exterior factors are handled as usual.

Thus

$$\begin{aligned} H_*\underline{bo}_{11} =& E\Big(e^{\circ 3}\circ z_{4i}\circ[\lambda^{-1}] : \alpha(i)\geq 4\Big)\\ &\otimes E\Big(\beta_{(i_1)}^{\circ 3}\circ\beta_{(i_2)}\circ\beta_{(i_3)}\circ\beta_{(i_4)}^{\circ 2}\circ\beta_{(i_5)}^{\circ 4}+\ldots :\\ &\quad i_5\geq i_4\geq i_3>i_2=1, i_1=0\Big)\\ &\otimes E\Big(\beta_{(i_1)}^{\circ 4}\circ\beta_{(i_2)}\circ\beta_{(i_3)}^{\circ 2}\circ\beta_{(i_4)}^{\circ 4}+\ldots : i_4\geq i_3\geq i_2>i_1\geq 0\Big)\\ &\otimes E\Big(\beta_{(i_1)}\circ\beta_{(i_2)}^{\circ 2}\circ\beta_{(i_3)}^{\circ 2}\circ\beta_{(i_4)}^{\circ 2}\circ\beta_{(i_5)}^{\circ 4}+\ldots :\\ &\quad i_5\geq i_4\geq i_3\geq i_2\geq i_1\geq 0\Big). \end{aligned}$$

We need to evaluate $\Theta(e^{\circ 3}\circ z_{4i}\circ[\lambda^{-1}])$ when $\alpha(i)=4$. By factoring $e^{\circ 3}\circ z_{4i}\circ[\lambda^{-1}]=e^{\circ 3}\circ(z_{4i}\circ[\lambda^{-1}])$, we see from step 8 that

$$\Theta(e^{\circ 3}\circ z_{4i}\circ[\lambda^{-1}])=\beta_{(0)}^{\circ 3}\circ\beta_{(i_2)}\circ\beta_{(i_3)}\circ\beta_{(i_4)}^{\circ 2}\circ\beta_{(i_5)}^{\circ 4}+\ldots$$

where we use the binary expansion $4i=2^{i_2}+2^{i_3}+2^{i_4+1}+2^{i_5+2}$, with $i_5\geq i_4\geq i_3>i_2\geq 2$.

2.2.1 The notations $A(s)$ and $C(n,k)$

Now we define some notation for the basic families of elements. We define

$$A(s)=\text{the set of all }\beta_{(i_1)}\circ\beta_{(i_2)}\circ\ldots\circ\beta_{(i_s)},\ \text{ for } s\geq 1,$$

$$A(0)=[1]$$

and inductively

$$C(n,k)=\text{the set of all }\beta_{(i_k)}\circ\beta_{(i_{k+1})}\circ\beta_{(i_{k+2})}^{\circ 2}\circ\beta_{(i_{k+3})}^{\circ 4}\circ C(n-1,k+4),$$

where $i_{k+1}>i_k\geq i_{k-1}+3$, starting from $C(0,k)=[1]$. We also assume implicitly that $i_m\geq i_n\geq 0$ whenever $m>n$.

Thus we rewrite

$$\begin{aligned} H_*\underline{bo}_{11} =& E\Big(e^{\circ 3}\circ z_{4i}\circ[\lambda^{-1}] : \alpha(i)\geq 4\Big) \\ &\otimes E\Big(\beta_{(i_1)}^{\circ 3}\circ\beta_{(i_2)}\circ\beta_{(i_3)}\circ\beta_{(i_4)}^{\circ 2}\circ\beta_{(i_5)}^{\circ 4}+\dots : i_3>i_2=1, i_1=0\Big) \\ &\otimes E\Big(\beta_{(i_1)}^{\circ 4}\circ\beta_{(i_2)}\circ\beta_{(i_3)}^{\circ 2}\circ\beta_{(i_4)}^{\circ 4}+\dots : i_2>i_1\Big) \\ &\otimes E\Big(A(1)\circ\beta_{(i_2)}^{\circ 2}\circ\beta_{(i_3)}^{\circ 2}\circ\beta_{(i_4)}^{\circ 2}\circ\beta_{(i_5)}^{\circ 4}+\dots\Big). \end{aligned}$$

STEP 12. To find $H_*\underline{bo}_{12}$: We use $H_*\underline{bo}_{11}$. Only the first factor of $H_*\underline{bo}_{11}$ requires any discussion. It gives rise to the factor

$$\Gamma\Big(\sigma(e^{\circ 3}\circ z_{4i}\circ[\lambda^{-1}]) : \alpha(i)\geq 4\Big)$$

in the E^2-term, which we rewrite as

$$E\Big(\gamma_{2^j}(\sigma(e^{\circ 3}\circ z_{4i}\circ[\lambda^{-1}])) : \alpha(i)\geq 4, j\geq 0\Big).$$

By step 4, $\gamma_{2^j}\Big(\sigma(e^{\circ 3}\circ z_{4i}\circ[\lambda^{-1}])\Big)$ is detected by the element written formally as $F^m\Big(\overline{z}_{2^j(8q+4)}\circ[\beta\lambda^{-2}]\Big)+\dots\in H_*\underline{bo}_{12}$, where as before we write $i=2^m(2q+1)-1$, so that $\alpha(i)=m+\alpha(q)\geq 4$.

We thus obtain

$$P\Big(\overline{z}_{4i}\circ[\beta\lambda^{-2}] : \alpha(i)\geq 5\Big)\otimes P\Big(F^j(\overline{z}_{4i}\circ[\beta\lambda^{-2}]) : \alpha(i)+j=5, i,j\geq 1\Big).$$

Therefore

$$\begin{aligned} H_*\underline{bo}_{12} =& P\Big(\overline{z}_{4i}\circ[\beta\lambda^{-2}] : \alpha(i)\geq 5\Big) \\ &\otimes P\Big(F^j(\overline{z}_{4i}\circ[\beta\lambda^{-2}])+\dots : \alpha(i)+j=5, i,j\geq 1\Big) \\ &\otimes E\Big(\beta_{(i_1)}^{\circ 4}\circ\beta_{(i_2)}\circ\beta_{(i_3)}\circ\beta_{(i_4)}^{\circ 2}\circ\beta_{(i_5)}^{\circ 4}+\dots : i_3>i_2=i_1+1\Big) \\ &\otimes E\Big(A(1)\circ\beta_{(i_2)}^{\circ 4}\circ\beta_{(i_3)}\circ\beta_{(i_4)}^{\circ 2}\circ\beta_{(i_5)}^{\circ 4}+\dots : i_3>i_2\Big) \\ &\otimes E\Big(A(2)\circ\beta_{(i_3)}^{\circ 2}\circ\beta_{(i_4)}^{\circ 2}\circ\beta_{(i_5)}^{\circ 2}\circ\beta_{(i_6)}^{\circ 4}+\dots\Big). \end{aligned}$$

From step 11,

$$\Theta\Big(e^{\circ 4} \circ z_{4i} \circ [\lambda^{-1}]\Big) = \beta_{(0)}^{\circ 4} \circ \beta_{(i_2)} \circ \beta_{(i_3)} \circ \beta_{(i_4)}^{\circ 2} \circ \beta_{(i_5)}^{\circ 4}$$

whenever $\alpha(i) = 4$, where we use the binary expansion

$$4i = 2^{i_2} + 2^{i_3} + 2^{i_4+1} + 2^{i_5+2},$$

so that $i_3 > i_2 \geq 2$. By step 4, we can rewrite

$$e^{\circ 4} \circ z_{4i} \circ [\lambda^{-1}] = F^m(\overline{z}_{8q+4} \circ [\beta\lambda^{-2}]) + \dots$$

where $i = 2^m(2q+1) - 1$, so that $\alpha(8q+4) + m = \alpha(q) + 1 + m = \alpha(i) + 1 = 5$. Since Θ commutes with V^j, we deduce that

$$\Theta\Big(F^m(\overline{z}_{2^j(8q+4)} \circ [\beta\lambda^{-2}]) + \dots\Big) = \beta_{(j)}^{\circ 4} \circ \beta_{(j+i_2)} \circ \beta_{(j+i_3)} \circ \beta_{(j+i_4)}^{\circ 2} \circ \beta_{(j+i_5)}^{\circ 4} + \dots$$

We reindex this as

$$\Theta\Big(F^j(\overline{z}_{4i} \circ [\beta\lambda^{-2}]) + \dots\Big) = \beta_{(i_1)}^{\circ 4} \circ \beta_{(i_2)} \circ \beta_{(i_3)} \circ \beta_{(i_4)}^{\circ 2} \circ \beta_{(i_5)}^{\circ 4} + \dots,$$

where 2^{i_1} is the largest power of 2 that divides i, which we have set equal to $2^{i_1}(2q+1)$, as above, and we use the binary expansion

$$2^j \cdot 4i - 2^{i_1+2} = 2^{i_2} + 2^{i_3} + 2^{i_4+1} + 2^{i_5+2},$$

where $i_3 > i_2 \geq i_1 + 2$. We note that $i_2 = i_1 + 2$ if $j > 0$, and that $i_2 \geq i_1 + 3$ if $j = 0$.

STEP 13. To find $H_*\underline{bo}_{13}$: We use $H_*\underline{bo}_{12}$.

By step 5, the factor

$$P\Big(\overline{z}_{4i} \circ [\beta\lambda^{-2}] + \dots : \alpha(i) \geq 5\Big)$$

in $H_*\underline{bo}_{12}$ yields the factor

$$E\Big(e \circ \overline{z}_{4i} \circ [\beta\lambda^{-2}] : \alpha(i) \geq 5\Big)$$

in $H_*\underline{bo}_{13}$.

The factor

$$P\Big(F^j(\overline{z}_{4i} \circ [\beta\lambda^{-2}]) + \dots : \alpha(i) + j = 5, i, j \geq 1\Big)$$

in $H_*\underline{bo}_{12}$ yields the factor

$$E\Big(\sigma(F^j(\overline{z}_{4i} \circ [\beta\lambda^{-2}])) : \alpha(i) + j = 5, i, j \geq 1\Big)$$

in the E^2-term. As in step 11, the generators map trivially to $H_*\underline{bo}_5$ and we must apply Θ instead. By step 12, the generator shown is detected by

$$\beta_{(0)} \circ \beta_{(i_1)}^{\circ 4} \circ \beta_{(i_2)} \circ \beta_{(i_3)} \circ \beta_{(i_4)}^{\circ 2} \circ \beta_{(i_5)}^{\circ 4},$$

where $i_3 > i_2 = i_1 + 2$ (since $j > 0$).

Thus,

$$\begin{aligned} H_*\underline{bo}_{13} =& E\Big(e \circ z_{4i} \circ [\beta\lambda^{-2}] : \alpha(i) \geq 5\Big) \\ & \otimes E\Big(\beta_{(i_1)} \circ \beta_{(i_2)}^{\circ 4} \circ \beta_{(i_3)} \circ \beta_{(i_4)} \circ \beta_{(i_5)}^{\circ 2} \circ \beta_{(i_6)}^{\circ 4} + \dots : \\ & \qquad i_4 > i_3 = i_2 + 2, i_1 = 0\Big) \\ & \otimes E\Big(A(1) \circ \beta_{(i_2)}^{\circ 4} \circ \beta_{(i_3)} \circ \beta_{(i_4)} \circ \beta_{(i_5)}^{\circ 2} \circ \beta_{(i_6)}^{\circ 4} + \dots : i_4 > i_3 = i_2 + 1\Big) \\ & \otimes E\Big(A(2) \circ \beta_{(i_3)}^{\circ 4} \circ \beta_{(i_4)} \circ \beta_{(i_5)}^{\circ 2} \circ \beta_{(i_6)}^{\circ 4} + \dots : i_4 > i_3\Big) \\ & \otimes E\Big(A(3) \circ \beta_{(i_4)}^{\circ 2} \circ \beta_{(i_5)}^{\circ 2} \circ \beta_{(i_6)}^{\circ 2} \circ \beta_{(i_7)}^{\circ 4} + \dots\Big). \end{aligned}$$

We need to know $\Theta(e \circ \overline{z}_{4i} \circ [\beta\lambda^{-2}])$ whenever $\alpha(i) = 5$. By decomposing within $H_*\underline{bo}_*$, we see from steps 5 and 8 that

$$\Theta(e \circ z_{4i} \circ [\beta\lambda^{-2}]) = \Big(\beta_{(0)} \circ \beta_{(i_2)}^{\circ 4}\Big) \circ \Big(\beta_{(i_3)} \circ \beta_{(i_4)} \circ \beta_{(i_5)}^{\circ 2} \circ \beta_{(i_6)}^{\circ 4}\Big) + \dots$$

where we use the binary expansion

$$4i = 2^{i_2+2} + 2^{i_3} + 2^{i_4} + 2^{i_5+1} + 2^{i_6+2},$$

so that $i_4 > i_3 \geq i_2 + 3$.

STEP 14. Now we can use everything we learned in step 6 to find $H_*\underline{bo}_{14}$.

Thus

$$\begin{aligned} H_*\underline{bo}_{14} = & E\Big(\overline{z}_{2i} \circ [\alpha^2\lambda^{-2}] : \alpha(i) \geq 6\Big) \\ & \otimes E\Big(\beta_{(i_1)}^{\circ 2} \circ \beta_{(i_2)}^{\circ 4} \circ \beta_{(i_3)} \circ \beta_{(i_4)} \circ \beta_{(i_5)}^{\circ 2} \circ \beta_{(i_6)}^{\circ 4} + \ldots : i_4 > i_3 = i_2 + 2\Big) \\ & \otimes E\Big(A(2) \circ \beta_{(i_3)}^{\circ 4} \circ \beta_{(i_4)} \circ \beta_{(i_5)} \circ \beta_{(i_6)}^{\circ 2} \circ \beta_{(i_7)}^{\circ 4} + \ldots : i_5 > i_4 = i_3 + 1\Big) \\ & \otimes E\Big(A(3) \circ \beta_{(i_4)}^{\circ 4} \circ \beta_{(i_5)} \circ \beta_{(i_6)}^{\circ 2} \circ \beta_{(i_7)}^{\circ 4} + \ldots : i_5 > i_4\Big) \\ & \otimes E\Big(A(4) \circ \beta_{(i_5)}^{\circ 2} \circ \beta_{(i_6)}^{\circ 2} \circ \beta_{(i_7)}^{\circ 2} \circ \beta_{(i_8)}^{\circ 4} + \ldots\Big). \end{aligned}$$

We need to know $\Theta(\overline{z}_{2i} \circ [\alpha^2\lambda^{-2}] + \ldots)$, whenever $\alpha(i) = 6$. Again we can factor within $H_*\underline{bo}_*$ and read off from step 6 and 8 that

$$\Theta(\overline{z}_{2i} \circ [\alpha^2\lambda^{-2}] + \ldots) = \beta_{(i_1)}^{\circ 2} \circ \beta_{(i_2)}^{\circ 4} \circ \beta_{(i_3)} \circ \beta_{(i_4)} \circ \beta_{(i_5)}^{\circ 2} \circ \beta_{(i_6)}^{\circ 4} + \ldots,$$

where we use the binary expansion

$$2i = 2^{i_1+1} + 2^{i_2+2} + 2^{i_3} + 2^{i_4} + 2^{i_5+1} + 2^{i_6+2},$$

so that $i_4 > i_3 \geq i_2 + 3$.

STEP 15. Now we use everything we learned in step 7 to find $H_*\underline{bo}_{15}$.

Thus

$$\begin{aligned} H_*\underline{bo}_{15} = & E\Big(\overline{z}_i \circ [\alpha\lambda^{-2}] : \alpha(i) \geq 7\Big) \\ & \otimes E\Big(A(1) \circ \beta_{(i_2)}^{\circ 2} \circ \beta_{(i_3)}^{\circ 4} \circ \beta_{(i_4)} \circ \beta_{(i_5)} \circ \beta_{(i_6)}^{\circ 2} \circ \beta_{(i_7)}^{\circ 4} + \ldots : \\ & \quad i_5 > i_4 = i_3 + 2\Big) \\ & \otimes E\Big(A(3) \circ \beta_{(i_4)}^{\circ 4} \circ \beta_{(i_5)} \circ \beta_{(i_6)} \circ \beta_{(i_7)}^{\circ 2} \circ \beta_{(i_8)}^{\circ 4} + \ldots : i_6 > i_5 = i_4 + 1\Big) \\ & \otimes E\Big(A(4) \circ \beta_{(i_5)}^{\circ 4} \circ \beta_{(i_6)} \circ \beta_{(i_7)}^{\circ 2} \circ \beta_{(i_8)}^{\circ 4} + \ldots : i_6 > i_5\Big) \\ & \otimes E\Big(A(5) \circ \beta_{(i_6)}^{\circ 2} \circ \beta_{(i_7)}^{\circ 2} \circ \beta_{(i_8)}^{\circ 2} \circ \beta_{(i_9)}^{\circ 4} + \ldots\Big). \end{aligned}$$

For $\alpha(i) = 7$ (by using steps 7 and 8), we have

$$\Theta(\overline{z}_i \circ [\alpha\lambda^{-2}]) = \beta_{(i_1)} \circ \beta_{(i_2)}^{\circ 2} \circ \beta_{(i_3)}^{\circ 4} \circ \beta_{(i_4)} \circ \beta_{(i_5)} \circ \beta_{(i_6)}^{\circ 2} \circ \beta_{(i_7)}^{\circ 4} + \ldots,$$

where we use the binary expansion

$$i = 2^{i_1} + 2^{i_2+1} + 2^{i_3+2} + 2^{i_4} + 2^{i_5} + 2^{i_6+1} + 2^{i_7+2},$$

so that $i_5 > i_4 \geq i_3 + 3$.

2.3 The structure theorem

The general pattern should now be apparent.

Theorem 2.3.1 *The Hopf ring $H_*\underline{bo}_*$ is a sub-Hopf ring of $H_*(\underline{KO}_* \times \underline{H}_*)$ and is the (graded) tensor product of the following four families of Hopf algebras:*

1. *Polynomial and exterior subalgebras of $H_*\underline{KO}_*$:*

$$P\Big(\overline{z}_i \circ [\lambda^{-n}] + \ldots : \alpha(i) \geq 4n\Big), \textit{ on } \underline{bo}_{8n}$$
$$P\Big(z_0 \circ [\lambda^{-n}], z_0^{-1} \circ [\lambda^{-n}]\Big), \textit{ on } \underline{bo}_{8n}, \textit{ for } n \leq 0$$
$$P\Big(e \circ z_{2i} \circ [\lambda^{-n}] : \alpha(i) \geq 4n\Big), \textit{ on } \underline{bo}_{8n+1}$$
$$P\Big(e^{\circ 2} \circ z_{4i} \circ [\lambda^{-n}] : \alpha(i) \geq 4n\Big), \textit{ on } \underline{bo}_{8n+2}$$
$$E\Big(e^{\circ 3} \circ z_{4i} \circ [\lambda^{-n}] : \alpha(i) \geq 4n\Big), \textit{ on } \underline{bo}_{8n+3}$$
$$P\Big(\overline{z}_{4i} \circ [\beta\lambda^{-(n+1)}] + \ldots : \alpha(i) \geq 4n+1\Big), \textit{ on } \underline{bo}_{8n+4}$$
$$P\Big(z_0 \circ [\beta\lambda^{-n}], z_0^{-1} \circ [\beta\lambda^{-n}]\Big), \textit{ on } \underline{bo}_{8n-4}, \textit{ for } n \leq 0$$
$$E\Big(e \circ z_{4i} \circ [\beta\lambda^{-(n+1)}] : \alpha(i) \geq 4n+1\Big), \textit{ on } \underline{bo}_{8n+5}$$
$$E\Big(\overline{z}_{2i} \circ [\alpha^2\lambda^{-(n+1)}] + \ldots : \alpha(i) \geq 4n+2\Big), \textit{ on } \underline{bo}_{8n+6}$$
$$E\Big([\alpha^2\lambda^{-n}] - 1\Big), \textit{ on } \underline{bo}_{8n-2}, \textit{ for } n \leq 0$$
$$E\Big(\overline{z}_i \circ [\alpha\lambda^{-(n+1)}] + \ldots : \alpha(i) \geq 4n+3\Big), \textit{ on } \underline{bo}_{8n+7}$$
$$E\Big([\alpha\lambda^{-n}] - 1\Big), \textit{ on } \underline{bo}_{8n-1}, \textit{ for } n \leq 0.$$

2. *Polynomial algebras on generators that decompose in $H_*\underline{KO}_*$, companions to the polynomial algebras in the first family:*

$$P\Big(F^j(\overline{z}_i \circ [\lambda^{-n}]) + \ldots : \alpha(i) + j = 4n, i, j \geq 1\Big), \textit{ on } \underline{bo}_{8n}$$
$$P\Big(F^j(e \circ z_{2i} \circ [\lambda^{-n}]) + \ldots : \alpha(i) + j = 4n, j \geq 1\Big), \textit{ on } \underline{bo}_{8n+1}$$
$$P\Big(F^j(e^{\circ 2} \circ z_{4i} \circ [\lambda^{-n}]) + \ldots : \alpha(i) + j = 4n, j \geq 1\Big), \textit{ on } \underline{bo}_{8n+2}$$
$$P\Big(F^j(\overline{z}_{4i} \circ [\beta\lambda^{-(n+1)}]) + \ldots : \alpha(i) + j = 4n+1, i, j \geq 1\Big), \textit{ on } \underline{bo}_{8n+4}.$$

3. *Exterior algebras involving $\beta_{(0)}$ that arise from the second family:*

$$E\Big(\beta_{(0)} \circ \beta^{\circ 2}_{(i_2)} \circ \beta^{\circ 2}_{(i_3)} \circ \beta^{\circ 4}_{(i_4)} \circ C(n,5) + \ldots\Big), \textit{ on } \underline{bo}_{8n+9}$$
$$E\Big(\beta^{\circ 3}_{(0)} \circ \beta_{(i_2)} \circ \beta^{\circ 2}_{(i_3)} \circ \beta^{\circ 4}_{(i_4)} \circ C(n,5) + \ldots : i_2 > 0\Big), \textit{ on } \underline{bo}_{8n+10}$$
$$E\Big(\beta^{\circ 3}_{(0)} \circ \beta_{(i_2)} \circ \beta_{(i_3)} \circ \beta^{\circ 2}_{(i_4)} \circ \beta^{\circ 4}_{(i_5)} \circ C(n,6) + \ldots : i_3 > i_2 = 1\Big), \textit{ on } \underline{bo}_{8n+11}$$
$$E\Big(\beta_{(0)} \circ \beta^{\circ 4}_{(i_2)} \circ \beta_{(i_3)} \circ \beta_{(i_4)} \circ \beta^{\circ 2}_{(i_5)} \circ \beta^{\circ 4}_{(i_6)} \circ C(n,7) + \ldots : i_4 > i_3 = i_2 + 2\Big), \textit{ on } \underline{bo}_{8n+13}.$$

4. *General exterior algebras that arise from the third family by unlimited suspension:*

$$E\Big(A(s) \circ \beta^{\circ 2}_{(i_{s+1})} \circ \beta^{\circ 2}_{(i_{s+2})} \circ \beta^{\circ 2}_{(i_{s+3})} \circ \beta^{\circ 4}_{(i_{s+4})} \circ C(n, s+5) + \ldots\Big)$$
$$E\Big(A(s) \circ \beta^{\circ 4}_{(i_{s+1})} \circ \beta_{(i_{s+2})} \circ \beta^{\circ 2}_{(i_{s+3})} \circ \beta^{\circ 4}_{(i_{s+4})} \circ C(n, s+5) + \ldots : i_{s+2} > i_{s+1}\Big)$$
$$E\Big(A(s) \circ \beta^{\circ 4}_{(i_{s+1})} \circ \beta_{(i_{s+2})} \circ \beta_{(i_{s+3})} \circ \beta^{\circ 2}_{(i_{s+4})} \circ \beta^{\circ 4}_{(i_{s+5})} \circ C(n, s+6) + \ldots : i_{s+3} > i_{s+2} = i_{s+1} + 1\Big)$$
$$E\Big(A(s) \circ \beta^{\circ 2}_{(i_{s+1})} \circ \beta^{\circ 4}_{(i_{s+2})} \circ \beta_{(i_{s+3})} \circ \beta_{(i_{s+4})} \circ \beta^{\circ 2}_{(i_{s+5})} \circ \beta^{\circ 4}_{(i_{s+6})} \circ C(n, s+7) + \ldots : i_{s+4} > i_{s+3} = i_{s+2} + 2\Big).$$

Proof. We have the result for $n \leq 1$. For $n \leq 0$, only the first family of Hopf algebras exists, and for $k \leq 0$, $H_*\underline{bo}_k = H_*\underline{KO}_k$. We note that there is no conflict between the many classes of generators we have exhibited. By induction, we assume that $H_*\underline{bo}_k$ is a subalgebra of $H_*\underline{KO}_k \otimes H_*\underline{H}_k$, as stated. The bar spectral sequence for $H_*\underline{bo}_{k+1}$ collapses, because the bar spectral sequence for $H_*(\underline{KO}_{k+1} \times \underline{H}_{k+1})$ is known to collapse; to see this, we only need to verify that the inclusion induces a monomorphism on the E^2-terms.

Each listed Hopf algebra in $H_*\underline{bo}_k$ gives rise to one or two Hopf algebras in $H_*\underline{bo}_{k+1}$. In $H_*\underline{bo}_k$ there is exactly one algebra from the first family. It is a subalgebra of $H_*\underline{KO}_k$, and we see from the steps already given how it gives rise to a first family algebra in $H_*\underline{bo}_{k+1}$, with the appropriate restriction on $\alpha(i)$. In addition, in half the cases it spawns a polynomial algebra in the second family, with the appropriate condition on $\alpha(i) + j$.

We treat each algebra of the third or fourth families in $H_*\underline{bo}_k$ as a subalgebra of $H_*\underline{H}_k$, and we have already noted how such an algebra gives rise to an algebra of the fourth family in $H_*\underline{bo}_{k+1}$.

The only case that requires any discussion is how an algebra from the second family in $H_*\underline{bo}_k$ gives rise to an algebra from the third family in $H_*\underline{bo}_{k+1}$. Because the generators decompose in $H_*\underline{KO}_k$, we must map them into $H_*\underline{H}_k$ instead. Thus we need to compute $\Theta\Big(F^j(\overline{z}_i \circ [\lambda^{-n}])\Big)$, etc.

We need $\Theta\Big(F^m(\overline{z}_k \circ [\lambda^{-n}])\Big)$ whenever $\alpha(k) + m = 4n$ and $m \geq 1$. We write $k = 2^j(2q+1)$, so that $\alpha(k) = \alpha(q) + 1$. By step 8, we have

$$V^j\Theta\Big(F^m(\overline{z}_k \circ [\lambda^{-n}])\Big) = \Theta(e \circ \overline{z}_i \circ [\alpha\lambda^{-n}]),$$

where

$$i = 2^m(2q+1) - 1 = (2^m - 1) + 2^{m+1}q,$$

so that i is odd and $\alpha(i) = \alpha(q) + m = 4n - 1$. We can evaluate this by factoring $e \circ \overline{z}_i \circ [\alpha\lambda^{-n}]$ in $H_*\underline{bo}_*$. Let

$$i = (2^{i_1} + 2^{i_2+1} + 2^{i_3+2}) + (2^{i_4} + 2^{i_5} + 2^{i_6+1} + 2^{i_7+2}) + \dots$$

$$+(2^{i_{4n-4}} + 2^{i_{4n-3}} + 2^{i_{4n-2}+1} + 2^{i_{4n-1}+2})$$

be the binary expansion of i, so that $i_1 = 0$, $i_5 > i_4 \geq i_3+3$, $i_9 > i_8 \geq i_7 + 3, \dots$ Then

$$e \circ \overline{z}_i \circ [\alpha\lambda^{-n}] = e \circ \Big(\overline{z}_{n_1} \circ [\alpha\lambda^{-1}]\Big) \circ \Big(\overline{z}_{n_2} \circ [\lambda^{-1}]\Big) \circ \dots \circ \Big(\overline{z}_{n_n} \circ [\lambda^{-1}]\Big),$$

where $n_1 = 2^{i_1} + 2^{i_2+1} + 2^{i_3+2}$, $n_2 = 2^{i_4} + 2^{i_5} + 2^{i_6+1} + 2^{i_7+2}$, etc. By steps 7 and 8,

$$\begin{aligned}\Theta(e \circ \overline{z}_i \circ [\alpha\lambda^{-n}]) &= \beta_{(0)} \circ \Big(\beta_{(0)} \circ \beta_{(i_2)}^{\circ 2} \circ \beta_{(i_3)}^{\circ 4}\Big) \circ \Big(\beta_{(i_4)} \circ \beta_{(i_5)} \circ \beta_{(i_6)}^{\circ 2} \circ \beta_{(i_7)}^{\circ 4}\Big) \circ \dots \\ &= \beta_{(0)}^{\circ 2} \circ \beta_{(i_2)}^{\circ 2} \circ \beta_{(i_3)}^{\circ 4} \circ c(n-1,4),\end{aligned}$$

where $c(n-1,4)$ denotes a typical generator of $C(n-1,4)$. Then

$$\Theta\Big(F^m(\overline{z}_k \circ [\lambda^{-n}])\Big) = \beta_{(j)}^{\circ 2} \circ \beta_{(j+i_2)}^{\circ 2} \circ \beta_{(j+i_3)}^{\circ 4} \circ c'(n-1,4) + \dots,$$

where $c'(n-1,4)$ denotes $c(n-1,4)$ with all indices raised by j.

In the bar spectral sequence for $H_*\underline{bo}_{8n+1}$, the polynomial algebra

$$P\Big(F^m(\overline{z}_k \circ [\lambda^{-n}]) + \dots : \alpha(k) + m = 4n\Big)$$

gives the factor

$$E\Big(\sigma(\beta_{(i_1)}^{\circ 2} \circ \beta_{(i_2)}^{\circ 2} \circ \beta_{(i_3)}^{\circ 4} \circ C(n-1,4))\Big)$$

in the E^2-term, which corresponds to

$$E\Big(\beta_{(0)} \circ \beta_{(i_2)}^{\circ 2} \circ \beta_{(i_3)}^{\circ 2} \circ \beta_{(i_4)}^{\circ 4} \circ C(n-1,5)\Big)$$

in $H_*\underline{bo}_{8n+1}$.

Next, we need $\Theta\Big(F^j(e \circ z_{2i} \circ [\lambda^{-n}])\Big)$ for $\alpha(i) + j = 4n, j \geq 1$. By step 1, $F^j(e \circ z_{2i}) = e \circ z_k$, where $k = 2^j(2i+1) - 1 = (2^j - 1) + 2^{j+1}i$, so that $\alpha(k) = j + \alpha(i) = 4n$. Again we factor $e \circ z_k \circ [\lambda^{-n}]$ in $H_*\underline{bo}_*$ and find

$$\Theta\Big(F^j(e \circ z_{2i} \circ [\lambda^{-n}])\Big) = \beta_{(0)} \circ \Big(\beta_{(i_1)} \circ \beta_{(i_2)} \circ \beta_{(i_3)}^{\circ 2} \circ \beta_{(i_4)}^{\circ 4}\Big) \circ \Big(\beta_{(i_5)} \circ \dots\Big) \circ \dots,$$

where we use the binary expansion

$$k = (2^{i_1} + 2^{i_2} + 2^{i_3+1} + 2^{i_4+2}) + (2^{i_5} + 2^{i_6} + 2^{i_7+1} + 2^{i_8+2}) + \dots$$

Since $j \geq 1, k$ is odd and $i_1 = 0$. We obtain the factor

$$E\Big(\beta_{(0)}^{\circ 3} \circ \beta_{(i_2)} \circ \beta_{(i_3)}^{\circ 2} \circ \beta_{(i_4)}^{\circ 4} \circ C(n-1, 5) + \dots : i_2 > 0\Big)$$

in $H_*\underline{bo}_{8n+2}$.

The case $\Theta\Big(F^j(e^{\circ 2} \circ z_{4i} \circ [\lambda^{-n}])\Big)$ is almost identical to the previous case, with indices doubled. We use $F^j(e^{\circ 2} \circ z_{4i}) = e^{\circ 2} \circ z_{2k}$, where k is odd as above. This time, to evaluate $\Theta(e^{\circ 2} \circ z_{2k} \circ [\lambda^{-n}])$, we need the binary expansion

$$2k = (2^{i_1} + 2^{i_2} + 2^{i_3+1} + 2^{i_4+2}) + (2^{i_5} + \dots) + \dots,$$

so that $i_1 = 1$ and $i_2 > 1$. We obtain the factor

$$E\Big(\beta_{(0)}^{\circ 3} \circ \beta_{(i_1)} \circ \beta_{(i_2)} \circ \beta_{(i_3)}^{\circ 2} \circ \beta_{(i_4)}^{\circ 4} \circ C(n-1, 5) + \dots : i_2 > i_1 = 1\Big)$$

in $H_*\underline{bo}_{8n+3}$.

Finally, we need $\Theta\Big(F^j(\overline{z}_{4i} \circ [\beta\lambda^{-(n+1)}])\Big)$ whenever $\alpha(i) + j = 4n + 1$ and $i, j \geq 1$. We reindex and note that by step 12

$$\begin{aligned} V^j\Theta\Big(F^m(\overline{z}_{2^j(8q+4)} \circ [\beta\lambda^{-(n+1)}]) + \dots\Big) &= \Theta\Big(F^m(\overline{z}_{8q+4} \circ [\beta\lambda^{-(n+1)}]) + \dots\Big) \\ &= \Theta\Big(e^{\circ 4} \circ z_{4i} \circ [\lambda^{-n}] + \dots\Big), \end{aligned}$$

where $i = 2^m(2q+1) - 1$, so that $\alpha(i) = m + \alpha(q) = 4n$. We factor $e^{\circ 4} \circ z_{4i} \circ [\lambda^{-n}]$ in $H_*\underline{bo}_*$ and use step 8 to obtain

$$\beta_{(0)}^{\circ 4} \circ \Big(\beta_{(i_1)} \circ \beta_{(i_2)} \circ \beta_{(i_3)}^{\circ 2} \circ \beta_{(i_4)}^{\circ 4}\Big) \circ c(n-1,5) + \ldots,$$

where we use the binary expansion

$$4i = (2^{i_1} + 2^{i_2} + 2^{i_3+1} + 2^{i_4+2}) + (2^{i_5} + \ldots) + \ldots$$

Since $m \geq 1$, i is odd and $i_2 > i_1 = 2$. Then

$$\begin{aligned}\Theta\Big(&F^m(\overline{z}_{2^j(8q+4)} \circ [\beta\lambda^{-(n+1)}]) + \ldots\Big) \\ &= \beta_{(j)}^{\circ 4} \circ \Big(\beta_{(j+i_1)} \circ \beta_{(j+i_2)} \circ \beta_{(j+i_3)}^{\circ 2} \circ \beta_{(j+i_4)}^{\circ 4}\Big) \circ c'(n-1,5) + \ldots\end{aligned}$$

and $H_*\underline{bo}_{8n+5}$ contains the factor (after reindexing)

$$E\Big(\beta_{(0)} \circ \beta_{(i_2)}^{\circ 4} \circ \beta_{(i_3)} \circ \beta_{(i_4)} \circ \beta_{(i_5)}^{\circ 2} \circ \beta_{(i_6)}^{\circ 4} \circ C(n-1,7) : i_4 > i_3 = i_2 + 2\Big).$$

Thus we have proven our hypothesis. ■

3 The Computation of the Hopf Rings $H_*\underline{bo\langle 4\rangle}_*$, $H_*\underline{bo\langle 2\rangle}_*$ and $H_*\underline{bo\langle 1\rangle}_*$

3.1 The computation of $H_*\underline{bo\langle 4\rangle}_*$

To calculate $H_*\underline{bo\langle 4\rangle}_*$, we keep in mind that we have the following exact triangle of spectra:

$$\Phi : \Sigma^3 K(\mathbf{Z}) \to \Sigma^8 bo \to bo\langle 4\rangle \to \Sigma^4 K(\mathbf{Z}).$$

We map $\Sigma^4 K(\mathbf{Z}) \to \Sigma^4 H = \Sigma^4 K(\mathbf{Z}/2)$ to simplify calculations.

Thus we have the maps

$$\zeta : H_*\underline{bo}_{n+8} \to H_*\underline{bo\langle 4\rangle}_n$$

and

$$\Theta : H_*\underline{bo\langle 4\rangle}_n \to H_*\underline{H}_{n+4}.$$

We note that all of the generators of $H_*\underline{bo\langle 4\rangle}_n$ either map non-trivially to $H_*\underline{H}_{n+4}$ or map from $H_*\underline{bo}_{n+8}$, but no element does both.

Our conclusion will be that the maps

$$\underline{bo\langle 4\rangle}_n \to \underline{bo}_n \to \underline{KO}_n$$

and

$$\Theta : \underline{bo\langle 4\rangle}_n \to \underline{H}_{n+4}$$

embed $H_*\underline{bo\langle 4\rangle}_n$ as a sub-Hopf algebra of $H_*(\underline{KO}_n \times \underline{H}_{n+4})$, which we describe.

We use

$$\Phi : \underline{bo}_0 \xrightarrow{\simeq} \underline{bo\langle 4\rangle}_{-8} = \underline{bo}_{-8}.$$

We also have the fibration

$$\Phi : \underline{bo}_4 \longrightarrow \underline{bo\langle 4\rangle}_{-4} \longrightarrow K(\mathbf{Z}, 0).$$

We keep in mind that $\underline{bo\langle 4\rangle}_{-4} = \underline{bo}_4 \times \mathbf{Z} = \underline{KO}_{-4}$. This is our starting place. Also, $\underline{bo\langle 4\rangle}_k = \underline{bo}_k$ for $k \leq -4$.

We recall that

$$H_*\underline{H}_0 = \mathbf{Z}[z_0, {z_0}^{-1}],$$

where $z_0 = [1]$ and ${z_0}^{-1} = [-1]$,

$$H_*\underline{bo}_4 = P\Big(\overline{z}_{4i} \circ [\beta\lambda^{-1}] : \alpha(i) \geq 1\Big),$$

and

$$H_*\underline{KO}_{-4} = P\Big(\overline{z}_{4i} \circ [\beta] : i > 0\Big) \otimes P\Big([\beta], [\beta]^{-1}\Big) = H_*\underline{bo\langle 4\rangle}_{-4}.$$

We clearly have

$$\zeta : \overline{z}_{4i} \circ [\beta\lambda^{-1}] \longmapsto \overline{z}_{4i} \circ [\beta]$$

for $i > 0$, and we also have the maps

$$\Theta([\beta]) = [1] \text{ and } \Theta([\beta]^{-1}) = [-1] = [1].$$

The general method of computation involves the same type of argument that we used in the proof of $H_*\underline{bo}_*$. Thus, we use the bar spectral sequence, properties of Hopf rings, and the properties of the Verschiebung and Frobenius maps. We must keep track of the map Θ to determine the structure of the exterior algebras in $H_*\underline{H}_*$, as we did in $H_*\underline{bo}_*$. We also use the map ζ to determine the structure of the sub-Hopf ring of $H_*\underline{bo\langle 4\rangle}_*$ in $H_*\underline{bo}_*$. We leave all proofs to the reader.

We define

$$A(s) = \text{the set of all } \beta_{(i_1)} \circ \beta_{(i_2)} \circ \ldots \circ \beta_{(i_s)}, \text{ for } s \geq 1,$$

$$A(0) = [1],$$

as before, but now we inductively define

$$C(n,k) = \text{the set of all } \beta^{\circ 4}_{(i_k)} \circ \beta_{(i_{k+1})} \circ \beta_{(i_{k+2})} \circ \beta^{\circ 2}_{(i_{k+3})} \circ C(n-1, k+4),$$

where $i_{k+2} > i_{k+1} \geq i_k + 3$, starting from $C(0,k) = [1]$. As in $H_* \underline{bo}_*$, we also assume implicitly that $i_m \geq i_n \geq 0$ whenever $m > n$.

Thus $H_* \underline{bo\langle 4\rangle}_*$ is the tensor product of the following four families of Hopf algebras:

1. Polynomial and exterior subalgebras of $H_* \underline{bo}_*$:

$$P\Big(\overline{z}_i \circ [\lambda^{-n}] + \ldots : \alpha(i) \geq 4n+3\Big), \text{on } \underline{bo\langle 4\rangle}_{8n}$$
$$P\Big(z_0 \circ [\lambda^{-n}], z_0^{-1} \circ [\lambda^{-n}]\Big), \text{on } \underline{bo\langle 4\rangle}_{8n}, \text{ for } n < 0$$
$$P\Big(e \circ z_{2i} \circ [\lambda^{-n}] : \alpha(i) \geq 4n+3\Big), \text{on } \underline{bo\langle 4\rangle}_{8n+1}$$
$$P\Big(e^{\circ 2} \circ z_{4i} \circ [\lambda^{-n}] : \alpha(i) \geq 4n+3\Big), \text{on } \underline{bo\langle 4\rangle}_{8n+2}$$
$$E\Big(e^{\circ 3} \circ z_{4i} \circ [\lambda^{-n}] : \alpha(i) \geq 4n+3\Big), \text{on } \underline{bo\langle 4\rangle}_{8n+3}$$
$$P\Big(\overline{z}_{4i} \circ [\beta\lambda^{-(n+1)}] + \ldots : \alpha(i) \geq 4n+4\Big), \text{on } \underline{bo\langle 4\rangle}_{8n+4}$$
$$P\Big(z_0 \circ [\beta\lambda^{-n}], z_0^{-1} \circ [\beta\lambda^{-n}]\Big), \text{on } \underline{bo\langle 4\rangle}_{8n-4}, \text{ for } n \leq 0$$
$$E\Big(e \circ z_{4i} \circ [\beta\lambda^{-(n+1)}] : \alpha(i) \geq 4n+4\Big), \text{on } \underline{bo\langle 4\rangle}_{8n+5}$$
$$E\Big(\overline{z}_{2i} \circ [\alpha^2\lambda^{-(n+1)}] + \ldots : \alpha(i) \geq 4n+5\Big), \text{on } \underline{bo\langle 4\rangle}_{8n+6}$$
$$E\Big([\alpha^2\lambda^{-n}] - 1\Big), \text{on } \underline{bo\langle 4\rangle}_{8n-2}, \text{ for } n < 0$$
$$E\Big(\overline{z}_i \circ [\alpha\lambda^{-(n+1)}] + \ldots : \alpha(i) \geq 4n+6\Big), \text{on } \underline{bo\langle 4\rangle}_{8n+7}$$
$$E\Big([\alpha\lambda^{-n}] - 1\Big), \text{on } \underline{bo\langle 4\rangle}_{8n-1}, \text{ for } n < 0.$$

2. Polynomial algebras on generators that decompose in $H_* \underline{bo}_*$, companions

to the polynomial algebras in the first family:

$$P\Big(F^j(\overline{z}_i \circ [\lambda^{-n}]) + \ldots : \alpha(i) + j = 4n+3, i, j \geq 1\Big), \text{on } \underline{bo\langle 4\rangle}_{8n}$$

$$P\Big(F^j(e \circ z_{2i} \circ [\lambda^{-n}]) + \ldots : \alpha(i) + j = 4n+3, j \geq 1\Big), \text{on } \underline{bo\langle 4\rangle}_{8n+1}$$

$$P\Big(F^j(e^{\circ 2} \circ z_{4i} \circ [\lambda^{-n}]) + \ldots : \alpha(i) + j = 4n+3, j \geq 1\Big), \text{on } \underline{bo\langle 4\rangle}_{8n+2}$$

$$P\Big(F^j(\overline{z}_{4i} \circ [\beta\lambda^{-(n+1)}]) + \ldots : \alpha(i) + j = 4n+4, i, j \geq 1\Big), \text{on } \underline{bo\langle 4\rangle}_{8n+4}.$$

3. Exterior algebras involving $\beta_{(0)}$ that arise from the second family:

$$E\Big(\beta_{(0)} \circ \beta_{(i_2)}^{\circ 2} \circ \beta_{(i_3)}^{\circ 2} \circ C(n,4) + \ldots\Big), \text{on } \underline{bo\langle 4\rangle}_{8n+1}$$

$$E\Big(\beta_{(0)}^{\circ 3} \circ \beta_{(i_2)} \circ \beta_{(i_3)}^{\circ 2} \circ C(n,4) + \ldots : i_2 > 0\Big), \text{on } \underline{bo\langle 4\rangle}_{8n+2}$$

$$E\Big(\beta_{(0)}^{\circ 3} \circ \beta_{(i_2)} \circ \beta_{(i_3)} \circ \beta_{(i_4)}^{\circ 2} \circ C(n,5) + \ldots : i_3 > i_2 = 1\Big), \text{on } \underline{bo\langle 4\rangle}_{8n+3}$$

$$E\Big(\beta_{(0)} \circ \beta_{(i_2)}^{\circ 4} \circ \beta_{(i_3)} \circ \beta_{(i_4)} \circ \beta_{(i_5)}^{\circ 2} \circ C(n,6) + \ldots : i_4 > i_3 = i_2 + 2\Big),$$
$$\text{on } \underline{bo\langle 4\rangle}_{8n+5}.$$

4. General exterior algebras that arise from the third family by unlimited suspension:

$$E\Big(A(s) \circ \beta_{(i_{s+1})}^{\circ 2} \circ \beta_{(i_{s+2})}^{\circ 2} \circ \beta_{(i_{s+3})}^{\circ 2} \circ C(n, s+4) + \ldots\Big)$$

$$E\Big(A(s) \circ \beta_{(i_{s+1})}^{\circ 4} \circ \beta_{(i_{s+2})} \circ \beta_{(i_{s+3})}^{\circ 2} \circ C(n, s+4) + \ldots : i_{s+2} > i_{s+1}\Big)$$

$$E\Big(A(s) \circ \beta_{(i_{s+1})}^{\circ 4} \circ \beta_{(i_{s+2})} \circ \beta_{(i_{s+3})} \circ \beta_{(i_{s+4})}^{\circ 2} \circ C(n, s+5) + \ldots :$$
$$i_{s+3} > i_{s+2} = i_{s+1} + 1\Big)$$

$$E\Big(A(s) \circ \beta_{(i_{s+1})}^{\circ 2} \circ \beta_{(i_{s+2})}^{\circ 4} \circ \beta_{(i_{s+3})} \circ \beta_{(i_{s+4})} \circ \beta_{(i_{s+5})}^{\circ 2} \circ C(n, s+6) + \ldots :$$
$$i_{s+4} > i_{s+3} = i_{s+2} + 2\Big).$$

3.2 The computation of $H_*\underline{bo\langle 2\rangle}_*$

To calculate $H_*\underline{bo\langle 2\rangle}_*$ we use the following exact triangle of spectra:

$$\Phi : \Sigma H \to bo\langle 4\rangle \to bo\langle 2\rangle \to \Sigma^2 H.$$

Once again we define

$$\Theta : H_*\underline{bo\langle 2\rangle}_n \to H_*\underline{H}_{n+2}$$

and

$$\zeta : H_*\underline{bo\langle 4\rangle}_n \to H_*\underline{bo\langle 2\rangle}_n.$$

The starting point is

$$\Phi : \underline{bo\langle 4\rangle}_{-2} \to \underline{bo\langle 2\rangle}_{-2} \to \underline{H}_0,$$

where we note that

$$H_*\underline{bo\langle 2\rangle}_{-2} = H_*\underline{KO}_{-2} = E\Big(\overline{z}_{2i} \circ [\alpha^2] : i > 0\Big) \otimes E\Big([\alpha^2] - 1\Big).$$

Also, $\underline{bo\langle 2\rangle}_k = \underline{bo}_k$ for $k \leq -2$. Thus we start with

$$\Theta\Big([\alpha^2] - 1\Big) = [1]$$

and

$$\zeta : \overline{z}_{2i} \circ [\alpha^2] \mapsto \overline{z}_{2i} \circ [\alpha^2],$$

for $i > 0$.

Again we define

$$A(s) = \text{ the set of all } \beta_{(i_1)} \circ \beta_{(i_2)} \circ \ldots \circ \beta_{(i_s)}, \text{ for } s \geq 1,$$

$$A(0) = [1],$$

but now we inductively define

$$C(n,k) = \text{the set of all } \beta^{\circ 2}_{(i_k)} \circ \beta^{\circ 4}_{(i_{k+1})} \circ \beta_{(i_{k+2})} \circ \beta_{(i_{k+3})} \circ C(n-1, k+4),$$

where $i_{k+3} > i_{k+2} \geq i_{k+1} + 3$, starting from $C(0,k) = [1]$.

Our conclusion is that the maps

$$\underline{bo\langle 2\rangle}_n \to \underline{bo}_n \to \underline{KO}_n$$

and

$$\Theta : \underline{bo\langle 2\rangle}_n \to \underline{H}_{n+2}$$

embed $H_*\underline{bo\langle 2\rangle}_n$ as a sub-Hopf algebra of $H_*(\underline{KO}_n \times \underline{H}_{n+2})$, which we describe. Thus $H_*\underline{bo\langle 2\rangle}_*$ is the tensor product of the following four families of Hopf algebras:

1. Polynomial and exterior subalgebras of $H_*\underline{bo}_*$:

$$P\Big(\overline{z}_i \circ [\lambda^{-n}] + \dots : \alpha(i) \geq 4n+2\Big), \text{on } \underline{bo\langle 2\rangle}_{8n}$$
$$P\Big(z_0 \circ [\lambda^{-n}], z_0^{-1} \circ [\lambda^{-n}]\Big), \text{on } \underline{bo\langle 2\rangle}_{8n}, \text{ for } n < 0$$
$$P\Big(e \circ z_{2i} \circ [\lambda^{-n}] : \alpha(i) \geq 4n+2\Big), \text{on } \underline{bo\langle 2\rangle}_{8n+1}$$
$$P\Big(e^{\circ 2} \circ z_{4i} \circ [\lambda^{-n}] : \alpha(i) \geq 4n+2\Big), \text{on } \underline{bo\langle 2\rangle}_{8n+2}$$
$$E\Big(e^{\circ 3} \circ z_{4i} \circ [\lambda^{-n}] : \alpha(i) \geq 4n+2\Big), \text{on } \underline{bo\langle 2\rangle}_{8n+3}$$
$$P\Big(\overline{z}_{4i} \circ [\beta\lambda^{-(n+1)}] + \dots : \alpha(i) \geq 4n+3\Big), \text{on } \underline{bo\langle 2\rangle}_{8n+4}$$
$$P\Big(z_0 \circ [\beta\lambda^{-n}], z_0^{-1} \circ [\beta\lambda^{-n}]\Big), \text{on } \underline{bo\langle 2\rangle}_{8n-4}, \text{ for } n \leq 0$$
$$E\Big(e \circ z_{4i} \circ [\beta\lambda^{-(n+1)}] : \alpha(i) \geq 4n+3\Big), \text{on } \underline{bo\langle 2\rangle}_{8n+5}$$
$$E\Big(\overline{z}_{2i} \circ [\alpha^2\lambda^{-(n+1)}] + \dots : \alpha(i) \geq 4n+4\Big), \text{on } \underline{bo\langle 2\rangle}_{8n+6}$$
$$E\Big([\alpha^2\lambda^{-n}] - 1\Big), \text{on } \underline{bo\langle 2\rangle}_{8n-2}, \text{ for } n \leq 0$$
$$E\Big(\overline{z}_i \circ [\alpha\lambda^{-(n+1)}] + \dots : \alpha(i) \geq 4n+5\Big), \text{on } \underline{bo\langle 2\rangle}_{8n+7}$$
$$E\Big([\alpha\lambda^{-n}] - 1\Big), \text{on } \underline{bo\langle 2\rangle}_{8n-1}, \text{ for } n < 0.$$

2. Polynomial algebras on generators that decompose in $H_*\underline{bo}_*$, companions

to the polynomial algebras in the first family:

$$P\Big(F^j(\overline{z}_i \circ [\lambda^{-n}]) + \ldots : \alpha(i) + j = 4n+2, i, j \geq 1\Big), \text{on } \underline{bo\langle 2\rangle}_{8n}$$

$$P\Big(F^j(e \circ z_{2i} \circ [\lambda^{-n}]) + \ldots : \alpha(i) + j = 4n+2, j \geq 1\Big), \text{on } \underline{bo\langle 2\rangle}_{8n+1}$$

$$P\Big(F^j(e^{\circ 2} \circ z_{4i} \circ [\lambda^{-n}]) + \ldots : \alpha(i) + j = 4n+2, j \geq 1\Big), \text{on } \underline{bo\langle 2\rangle}_{8n+2}$$

$$P\Big(F^j(\overline{z}_{4i} \circ [\beta\lambda^{-(n+1)}]) + \ldots : \alpha(i) + j = 4n+3, i, j \geq 1\Big), \text{on } \underline{bo\langle 2\rangle}_{8n+4}.$$

3. Exterior algebras involving $\beta_{(0)}$ that arise from the second family:

$$E\Big(\beta_{(0)} \circ \beta_{(i_2)}^{\circ 2} \circ C(n,3) + \ldots\Big), \text{on } \underline{bo\langle 2\rangle}_{8n+1}$$

$$E\Big(\beta_{(0)}^{\circ 3} \circ \beta_{(i_2)} \circ C(n,3) + \ldots : i_2 > 0\Big), \text{on } \underline{bo\langle 2\rangle}_{8n+2}$$

$$E\Big(\beta_{(0)}^{\circ 3} \circ \beta_{(i_2)} \circ \beta_{(i_3)} \circ C(n,4) + \ldots : i_3 > i_2 = 1\Big), \text{on } \underline{bo\langle 2\rangle}_{8n+3}$$

$$E\Big(\beta_{(0)} \circ \beta_{(i_2)}^{\circ 4} \circ \beta_{(i_3)} \circ \beta_{(i_4)} \circ C(n,5) + \ldots : i_4 > i_3 = i_2 + 2\Big), \text{on } \underline{bo\langle 2\rangle}_{8n+5}.$$

4. General exterior algebras that arise from the third family by unlimited suspension:

$$E\Big(A(s) \circ \beta_{(i_{s+1})}^{\circ 2} \circ \beta_{(i_{s+2})}^{\circ 2} \circ C(n, s+3) + \ldots\Big)$$

$$E\Big(A(s) \circ \beta_{(i_{s+1})}^{\circ 4} \circ \beta_{(i_{s+2})} \circ C(n, s+3+\ldots) : i_{s+2} > i_{s+1}\Big)$$

$$E\Big(A(s) \circ \beta_{(i_{s+1})}^{\circ 4} \circ \beta_{(i_{s+2})} \circ \beta_{(i_{s+3})} \circ C(n, s+4) + \ldots : i_{s+3} > i_{s+2} = i_{s+1} + 1\Big)$$

$$E\Big(A(s) \circ \beta_{(i_{s+1})}^{\circ 2} \circ \beta_{(i_{s+2})}^{\circ 4} \circ \beta_{(i_{s+3})} \circ \beta_{(i_{s+4})} \circ C(n, s+5) + \ldots : i_{s+4} > i_{s+3} = i_{s+2} + 2\Big).$$

3.3 The computation of $H_*\underline{bo\langle 1\rangle}_*$

To calculate $H_*\underline{bo\langle 1\rangle}_*$ we keep in mind that we have the following exact triangle of spectra:

$$\Phi : H \to bo\langle 2\rangle \to bo\langle 1\rangle \to \Sigma^1 H.$$

We again define

$$\Theta : H_*\underline{bo\langle 1\rangle}_n \to H_*\underline{H}_{n+1}$$

and

$$\zeta : H_*\underline{bo\langle 2\rangle}_n \to H_*\underline{bo\langle 1\rangle}_n.$$

The starting point is

$$\Phi : \underline{bo\langle 2\rangle}_{-1} \to \underline{bo\langle 1\rangle}_{-1} \to \underline{H}_0,$$

where we note that

$$H_*\underline{bo\langle 1\rangle}_{-1} = H_*\underline{KO}_{-1} = E\Big(\overline{z}_i \circ [\alpha] : i > 0\Big) \otimes E\Big([\alpha] - 1\Big).$$

Also, $\underline{bo\langle 1\rangle}_k = \underline{bo}_k$ for $k \leq -1$.

We define

$$A(s) = \text{the set of all } \beta_{(i_1)} \circ \beta_{(i_2)} \circ \ldots \circ \beta_{(i_s)}, \text{ for } s \geq 1,$$

$$A(0) = [1]$$

as before, but now we inductively define

$$C(n,k) = \text{the set of all } \beta_{(i_k)} \circ \beta^{\circ 2}_{(i_{k+1})} \circ \beta^{\circ 4}_{(i_{k+2})} \circ \beta_{(i_{k+3})} \circ C(n-1, k+4),$$

where $i_{k+3} \geq i_{k+2} + 3$ and $i_k > i_{k-1}$, starting from $C(0,k) = [1]$.

Our conclusion is that the maps

$$\underline{bo\langle 1\rangle}_n \to \underline{bo}_n \to \underline{KO}_n$$

and

$$\Theta : \underline{bo\langle 1\rangle}_n \to \underline{H}_{n+1}$$

embed $H_*\underline{bo\langle 1\rangle}_n$ as a sub-Hopf algebra of $H_*(\underline{KO}_n \times \underline{H}_{n+1})$, which we describe. Thus $H_*\underline{bo\langle 1\rangle}_*$ is the tensor product of the following four families of Hopf algebras:

1. Polynomial and exterior subalgebras of $H_*\underline{bo}_*$:

$$P\Big(\overline{z}_i \circ [\lambda^{-n}] + \ldots : \alpha(i) \geq 4n+1\Big), \text{on } \underline{bo\langle 1\rangle}_{8n}$$
$$P\Big(z_0 \circ [\lambda^{-n}], z_0^{-1} \circ [\lambda^{-n}]\Big), \text{on } \underline{bo\langle 1\rangle}_{8n}, \text{ for } n < 0$$
$$P\Big(e \circ z_{2i} \circ [\lambda^{-n}] : \alpha(i) \geq 4n+1\Big), \text{on } \underline{bo\langle 1\rangle}_{8n+1}$$
$$P\Big(e^{\circ 2} \circ z_{4i} \circ [\lambda^{-n}] : \alpha(i) \geq 4n+1\Big), \text{on } \underline{bo\langle 1\rangle}_{8n+2}$$
$$E\Big(e^{\circ 3} \circ z_{4i} \circ [\lambda^{-n}] : \alpha(i) \geq 4n+1\Big), \text{on } \underline{bo\langle 1\rangle}_{8n+3}$$
$$P\Big(\overline{z}_{4i} \circ [\beta\lambda^{-(n+1)}] + \ldots : \alpha(i) \geq 4n+2\Big), \text{on } \underline{bo\langle 1\rangle}_{8n+4}$$
$$P\Big(z_0 \circ [\beta\lambda^{-n}], z_0^{-1} \circ [\beta\lambda^{-n}]\Big), \text{on } \underline{bo\langle 1\rangle}_{8n-4}, \text{ for } n \leq 0$$
$$E\Big(e \circ z_{4i} \circ [\beta\lambda^{-(n+1)}] : \alpha(i) \geq 4n+2\Big), \text{on } \underline{bo\langle 1\rangle}_{8n+5}$$
$$E\Big(\overline{z}_{2i} \circ [\alpha^2\lambda^{-(n+1)}] + \ldots : \alpha(i) \geq 4n+3\Big), \text{on } \underline{bo\langle 1\rangle}_{8n+6}$$
$$E\Big([\alpha^2\lambda^{-n}] - 1\Big), \text{on } \underline{bo\langle 1\rangle}_{8n-2}, \text{ for } n \leq 0$$
$$E\Big(\overline{z}_i \circ [\alpha\lambda^{-(n+1)}] + \ldots : \alpha(i) \geq 4n+4\Big), \text{on } \underline{bo\langle 1\rangle}_{8n+7}$$
$$E\Big([\alpha\lambda^{-n}] - 1\Big), \text{on } \underline{bo\langle 1\rangle}_{8n-1}, \text{ for } n \leq 0.$$

2. Polynomial algebras on generators that decompose in $H_*\underline{bo}_*$, companions to the polynomial algebras in the first family:

$$P\Big(F^j(\overline{z}_i \circ [\lambda^{-n}]) + \ldots : \alpha(i) + j = 4n+1, i, j \geq 1\Big), \text{on } \underline{bo\langle 1\rangle}_{8n}$$
$$P\Big(F^j(e \circ z_{2i} \circ [\lambda^{-n}]) + \ldots : \alpha(i) + j = 4n+1, j \geq 1\Big), \text{on } \underline{bo\langle 1\rangle}_{8n+1}$$
$$P\Big(F^j(e^{\circ 2} \circ z_{4i} \circ [\lambda^{-n}]) + \ldots : \alpha(i) + j = 4n+1, j \geq 1\Big), \text{on } \underline{bo\langle 1\rangle}_{8n+2}$$
$$P\Big(F^j(\overline{z}_{4i} \circ [\beta\lambda^{-(n+1)}]) + \ldots : \alpha(i) + j = 4n+2, i, j \geq 1\Big), \text{on } \underline{bo\langle 1\rangle}_{8n+4}.$$

3. Exterior algebras involving $\beta_{(0)}$ that arise from the second family:

$$E\Big(\beta_{(0)}^{\circ 3} \circ C(n,2) + \dots\Big), \text{on } \underline{bo\langle 1\rangle}_{8n+2}$$
$$E\Big(\beta_{(0)}^{\circ 3} \circ \beta_{(i_2)} \circ C(n,3) + \dots : i_2 = 1\Big), \text{on } \underline{bo\langle 1\rangle}_{8n+3}$$
$$E\Big(\beta_{(0)} \circ \beta_{(i_2)}^{\circ 4} \circ \beta_{(i_3)} \circ C(n,4) + \dots : i_3 = i_2 + 2\Big), \text{on } \underline{bo\langle 1\rangle}_{8n+5}$$
$$E\Big(\beta_{(0)} \circ \beta_{(i_2)}^{\circ 2} \circ \beta_{(i_3)}^{\circ 2} \circ \beta_{(i_4)}^{\circ 4} \circ \beta_{(i_5)} \circ C(n,6) + \dots : i_5 \geq i_4 + 3\Big), \text{on } \underline{bo\langle 1\rangle}_{8n+9}.$$

4. General exterior algebras that arise from the third family by unlimited suspension:

$$E\Big(A(s) \circ \beta_{(i_{s+1})}^{\circ 4} \circ C(n, s+2) + \dots\Big)$$
$$E\Big(A(s) \circ \beta_{(i_{s+1})}^{\circ 4} \circ \beta_{(i_{s+2})} \circ C(n, s+3) + \dots : i_{s+2} = i_{s+1} + 1\Big)$$
$$E\Big(A(s) \circ \beta_{(i_{s+1})}^{\circ 2} \circ \beta_{(i_{s+2})}^{\circ 4} \circ \beta_{(i_{s+3})} \circ C(n, s+4) + \dots : i_{s+3} = i_{s+2} + 2\Big)$$
$$E\Big(A(s) \circ \beta_{(i_{s+1})}^{\circ 2} \circ \beta_{(i_{s+2})}^{\circ 2} \circ \beta_{(i_{s+3})}^{\circ 2} \circ \beta_{(i_{s+4})}^{\circ 4} \circ \beta_{(i_{s+5})} \circ C(n, s+6) + \dots : i_{s+5} \geq i_{s+4} + 3\Big).$$

3.4 The computation of $H_*\underline{bo}_*$

We note that we can use this same process to come full circle, calculating $H_*\underline{bo}_*$ from $H_*\underline{bo\langle 1\rangle}_*$. Here we start with the exact triangle of spectra

$$\Phi : \Sigma^{-1}K(\mathbf{Z}) \to bo\langle 1\rangle \to bo\langle 0\rangle \to K(\mathbf{Z}),$$

and proceed from this point as we did in the three previous calculations. ■

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