

The  $E(1, 2)$  Cohomology of the Eilenberg-Mac  
Lane Space  $K(\mathbb{Z}, 3)$

by  
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## Abstract

In this paper, we give new bases of  $E_* \left( \underline{BP} \langle 1 \rangle_4 \right)$  and  $E_* \left( \underline{BP} \langle 1 \rangle_6 \right)$  in order to define a basis of  $E^* \left( \underline{BP} \langle 1 \rangle_4 \right)$  and  $E^* \left( \underline{BP} \langle 1 \rangle_6 \right)$ . With these two new bases, we have a short exact sequence of Hopf algebras

$$E^* \longleftarrow E^* (K(\mathbb{Z}, 3)) \xleftarrow{f^*} E^* [[y_{(i)} : i \geq 0]] \xleftarrow{v_1^*} E^* [[x_{(i)} : i \geq 0]] \longleftarrow E^*.$$

The action of  $v_1^*$  is calculated by using formal group laws and used to compute  $E^* (K(\mathbb{Z}, 3))$  as a completion of  $E^* [z_{(0)}]$ .

Readers: Dr. J. Michael Boardman (Advisor) and Dr. Jack Morava

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# 1 Introduction

The Eilenberg-Mac Lane Space  $K(\mathbb{Z}, 2)$  is readily recognized as complex projective space  $\mathbb{C}P^\infty$ . Given any multiplicative complex-oriented generalized cohomology theory  $E^*(-)$ , a classical result of Dold states that

$$E^*(\mathbb{C}P^\infty) = E^*[[x]],$$

the graded ring of formal power series over the coefficient ring  $E^*$  of  $E$ , where  $x$  denotes the Chern class of the Hopf line bundle over  $\mathbb{C}P^\infty$ .

This paper studies the  $E$ -cohomology of the next Eilenberg-Mac Lane space  $K(\mathbb{Z}, 3)$ , which may be viewed as the classifying space of  $K(\mathbb{Z}, 2)$ , for certain complex-oriented  $E$ .

Note that, for any space  $X$ , a cohomology class in  $E^k(X)$  corresponds to a map  $X \rightarrow \underline{E}_k$ , defined up to homotopy, where  $\underline{E}_k$  comes from the Brown Representation Theorem, which can be stated as: for any cohomology theory  $E$ , for all  $k$ , there exists a classifying space  $\underline{E}_k$  such that  $E^k(X) \cong [X, \underline{E}_k]$ , the set of homotopy classes of unbased maps  $X \rightarrow \underline{E}_k$ .

For a prime  $p$ , the Johnson-Wilson spectrum  $BP\langle q \rangle$  is a  $BP$ -module spectrum with  $\pi_*(BP\langle q \rangle) = \mathbb{Z}_{(p)}[v_1, \dots, v_q]$ . For  $q > 0$ , there is a stable cofibre sequence

$$\Sigma^{2(p^q-1)}BP\langle q \rangle \xrightarrow{v_q} BP\langle q \rangle \rightarrow BP\langle q-1 \rangle \rightarrow \Sigma^{2p^q-1}BP\langle q \rangle.$$

Unstably, the boundary map is

$$\underline{BP}\langle k-1 \rangle_j \rightarrow \underline{BP}\langle k \rangle_{j+2p^k-1}$$

and the iteration is

$$K(\mathbb{Z}/p^i, q+1) \rightarrow K(\mathbb{Z}_{(p)}, q+2) \rightarrow \underline{BP}\langle 1 \rangle_{q+2p+1} \rightarrow \dots \rightarrow \underline{BP}\langle q \rangle_{g(q)}$$

where  $g(q) = 2(p^q + p^{q-1} + \dots + p + 1)$  and  $\underline{BP}\langle 0 \rangle_{q+2} = K(\mathbb{Z}_{(p)}, q+2)$ . In [RWY], there is an exact sequence in the category of  $K(n)_*$ -Hopf algebras:

$$\begin{aligned} K(n)_* &\rightarrow K(n)_*(K(\mathbb{Z}_{(p)}, q+2)) \\ &\rightarrow K(n)_*\left(\underline{BP}\langle q \rangle_{g(q)}\right) \xrightarrow{v_{q*}} K(n)_*\left(\underline{BP}\langle q \rangle_{g(q)-|v_q|}\right). \end{aligned}$$

This induces, by Theorem 1.19 in [RWY],

$$BP^* (K (\mathbb{Z}_{(p)}, q + 2)) \cong \frac{BP^* \left( \underline{BP} \langle q \rangle_{g(q)} \right)}{(v_q^*)},$$

where  $(v_q^*)$  denotes the ideal generated by the image under  $v_q^*$  of the augmentation ideal in  $BP^* \left( \underline{BP} \langle q \rangle_{g(q)-|v_q|} \right)$ .

In 1999, Tamanoi [T], who was motivated by the above result, created new generators and wrote down a very simple relation in terms of his new generators. Let  $\iota_{g(q)+t}^{\langle q \rangle}$  be the inclusion map

$$\iota_{g(q)+t}^{\langle q \rangle} : \underline{BP} \langle q \rangle_{g(q)+t} \longrightarrow \underline{BP}_{g(q)+t}$$

for  $t \leq 0$ . Recall that by Wilson's Splitting Theorem [Wil75],  $\underline{BP} \langle q \rangle_{g(q)+t}$  is a factor space of  $\underline{BP}_{g(q)+t}$ . Thus,  $\iota_{g(q)+t}^{\langle q \rangle}$  exists. This map defines a cohomology class

$$\iota_{g(q)+t}^{\langle q \rangle} \in BP^{g(q)+t} \left( \underline{BP} \langle q \rangle_{g(q)+t} \right),$$

namely the  $BP$  fundamental class for  $BP^{g(q)+t} \left( \underline{BP} \langle q \rangle_{g(q)+t} \right)$ . Then, Tamanoi defined the  $v_q$ -series

$$[v_q] = v_q^* \left( \iota_{g(q)}^{\langle q \rangle} \right) \in BP^{g(q)-|v_q|} \left( \underline{BP} \langle q \rangle_{g(q)} \right),$$

which comes from the composition map

$$\underline{BP} \langle q \rangle_{g(q)} \xrightarrow{v_q} \underline{BP} \langle q \rangle_{g(q)-|v_q|} \xrightarrow{\iota_{g(q)-|v_q|}^{\langle q \rangle}} \underline{BP}_{g(q)-|v_q|}.$$

Also, he showed that

$$[v_q] = v_q \theta_{q,q}^{\langle q \rangle} + v_{q+1} \theta_{q+1,q}^{\langle q \rangle} + \cdots + v_{q+j} \theta_{q+j,q}^{\langle q \rangle} + \cdots,$$

where the class  $\theta_{q+j,q}^{\langle q \rangle}$  with  $j \geq 1$  is defined as the following composition:

$$\begin{aligned} \underline{BP} \langle q \rangle_{g(q)} &\longrightarrow \underline{BP}_{g(q)} \xrightarrow{v_q} \underline{BP}_{g(q)-|v_q|} \xrightarrow{\text{proj}^{\langle q+j \rangle}} \underline{BP} \langle q+j \rangle_{g(q)+2p^{q+j}-2p^q} \\ &\longrightarrow \underline{BP}_{g(q)+2p^{q+j}-2p^q} \end{aligned}$$

and the projection is provided by Wilson's theorem.

We take  $p = 2$ . By the short exact sequence of Hopf algebras

$$BP^* \longleftarrow BP^*(K(\mathbb{Z}, 3)) \xleftarrow{f^*} BP^*(\underline{BP}\langle 1 \rangle_6) \xleftarrow{v_1^*} BP^*(\underline{BP}\langle 1 \rangle_4) \longleftarrow BP^*,$$

we have

$$BP^*(K(\mathbb{Z}, 3)) \cong \frac{BP^*(\underline{BP}\langle 1 \rangle_6)}{(v_1^*)},$$

where  $(v_1^*)$  denotes the image under  $v_1^*$  of the augmentation ideal of  $BP^*(\underline{BP}\langle 1 \rangle_4)$ .

The relation Tamanoi found in this case is

$$v_1^*(x_{(0)}) = v_1 y_{(0)} + v_2 y_{(1)} + \cdots + v_i y_{(i-1)} + \cdots.$$

By tensoring with  $E(1, 2)^*$ , we have  $E(1, 2)^*(K(\mathbb{Z}, 3)) \cong E(1, 2)^*(\underline{BP}\langle 1 \rangle_6) / (v_1^*)$  where  $v_1^*(x_{(0)}) = v_1 y_{(0)} + v_2 y_{(1)}$ .

We use the formal group law to create a new set of generators  $y'_{(i)}$ , for which the relations in  $E(1, 2)^*(\underline{BP}\langle 1 \rangle_6)$  take the simplified form

$$v_1^*(x_{(i)}) = v_1^{2^i} y'_{(i)} + v_2^{2^i} y'_{(i+1)} + D_{(i)}$$

for  $i \geq 0$ , where  $D_{(i)}$  is decomposable.

This paper is organized as follows: In section 2, we collect some basic notation and theorems for the cohomology theories  $BP$  and  $E(k, n)$ . We calculate some formal group laws in section 3. In section 4, a basis of  $BP_*(\underline{BP}\langle 1 \rangle_4)$  and  $BP_*(\underline{BP}\langle 1 \rangle_6)$  will be given so that we can write down a dual basis for  $BP^*(\underline{BP}\langle 1 \rangle_4)$  and  $BP^*(\underline{BP}\langle 1 \rangle_6)$ . These are used to prove the main results.

## 2 Preliminary

The coefficient ring for Brown-Peterson cohomology is

$$BP^* \cong \mathbb{Z}_{(p)}[v_1, v_2, \dots]$$

where the degree of  $v_n$  is  $-2(p^n - 1)$ . In homology, we often write the coefficient ring as  $BP_*$ , with  $BP_n = BP^{-n}$  (see [RW77]).

We have the Hopf algebroid  $(BP_*, BP_*BP)$  which satisfies:

1.  $BP_* \cong \mathbb{Z}_{(p)} [v_1, v_2, \dots]$ , where  $\deg v_n = 2(p^n - 1)$ .
2.  $BP_*BP \cong BP_* [t_1, t_2, \dots]$ , where  $\deg t_n = 2(p^n - 1)$ .
3.  $\eta_R : BP_* \rightarrow BP_*BP$  is a ring homomorphism, where  $\eta_R v$  is written as  $[v]$ .

Let  $I_n$  be the ideal  $(p, v_1, v_2, \dots, v_{n-1})$ . And,  $v_0 = p$ .

Let  $BP \langle n \rangle_* = \mathbb{Z}_{(p)} [v_1, v_2, \dots, v_n]$  and  $\left\{ \underline{BP \langle n \rangle_*} \right\}$  be the  $\Omega$ -spectrum of  $BP \langle n \rangle$ .

Let  $P(n)$  be the theory with coefficient ring

$$P(n)^* \cong BP^* / I_n.$$

And,

$$P(0) = BP \text{ or } BP_p^\wedge$$

where  $BP_p^\wedge$  is the  $p$ -adic completion of  $BP$ . We have stable cofibrations

$$\Sigma^{2(p^n-1)} P(n) \xrightarrow{v_n} P(n) \longrightarrow P(n+1)$$

which lead to long exact sequences in cohomology.

There are spectra  $E(k, n)$  with coefficient rings

$$E(k, n)^* \cong v_n^{-1} BP \langle n \rangle^* / I_k$$

with similar stable cofibrations

$$\Sigma^{2(p^k-1)} E(k, n) \xrightarrow{v_k} E(k, n) \longrightarrow E(k+1, n)$$

and long exact sequences. In particular, for  $1 \leq k \leq n$ ,

$$E(k, n)^* \cong v_n^{-1} BP \langle n \rangle^* / I_k = \mathbb{F}_p [v_k, v_{k+1}, \dots, v_n, v_n^{-1}].$$

A special case, when  $k = n > 0$ , is the  $n^{\text{th}}$  Morava  $K$ -theory,  $K(n)^*(X)$ , with

$$K(n)^* \cong \mathbb{Z}/p [v_n, v_n^{-1}].$$

Also, when  $k = 0$ ,  $E(n)^* = E(0, n)^* \cong v_n^{-1} BP \langle n \rangle^* = \mathbb{Z}_{(p)} [v_1, v_2, \dots, v_n, v_n^{-1}]$ .

Let  $E$  denote any of  $BP_p^\wedge$ ,  $P(n)$  or  $E(k, n)$  where  $0 \leq k \leq n$  and  $0 < n$ .

## 2.1 Formal Group Laws

We summarize the properties of formal group laws. We are working with the Brown-Peterson spectrum with prime 2.

**Definition 2.1** *A formal group law is a power series*

$$F(y, z) = \sum_{i, j \geq 0} a_{ij} y^i z^j$$

such that

1.  $F(y, 0) = y$  and  $F(0, z) = z$ ,
2.  $F(y, F(z, w)) = F(F(y, z), w)$ ,

where  $a_{ij}$  is in a commutative ring with unit. Moreover, we say a formal group law is commutative if  $F(y, z) = F(z, y)$ . We write  $y +_F z$  for  $F(y, z)$ .

**Example 2.2** *We have two trivial examples of commutative formal group laws:*

$$F(y, z) = y + z$$

and

$$F(y, z) = y + z + yz.$$

By the first condition in the formal group law definition, we have

$$a_{i0} = \begin{cases} 1 & , \text{ if } i = 1 \\ 0 & , \text{ if } i \neq 1 \end{cases}$$

and

$$a_{0j} = \begin{cases} 1 & , \text{ if } j = 1 \\ 0 & , \text{ if } j \neq 1 \end{cases}.$$

Therefore, we may write our formal group law in the following form

$$F(y, z) = y + z + \sum_{i, j \geq 1} a_{ij} y^i z^j$$

Let  $x \in BP^2(\mathbb{C}P^\infty)$  be the Conner-Floyd Chern class.

**Proposition 2.3** (In [RW77] Lemma 3.3 on Page 255) *We have*

1.  $BP^*(\mathbb{C}P^\infty) \cong BP^*[[x]]$ , the ring of formal power series on  $x = x^{\mathbb{C}P^\infty}$  over  $BP^*$ .
2.  $BP^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty) \cong BP^*(\mathbb{C}P^\infty) \widehat{\otimes}_{BP^*} BP^*(\mathbb{C}P^\infty)$ .
3.  $BP_*(\mathbb{C}P^\infty)$  is  $BP_*$ -free on  $\beta_i \in BP_{2i}(\mathbb{C}P^\infty)$  for  $i \geq 0$ , dual to  $(x^{\mathbb{C}P^\infty})^i$ , i.e.  $\langle (x^{\mathbb{C}P^\infty})^i, \beta_j \rangle = \delta_{ij}$ .
4.  $BP_*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty) \simeq BP_*(\mathbb{C}P^\infty) \otimes_{BP_*} BP_*(\mathbb{C}P^\infty)$ .
5. The diagonal  $\mathbb{C}P^\infty \rightarrow \mathbb{C}P^\infty \times \mathbb{C}P^\infty$  induces a coproduct  $\psi$  on  $BP_*(\mathbb{C}P^\infty)$  with  $\psi(\beta_n) = \sum_{i=0}^n \beta_i \otimes \beta_{n-i}$ .
6. The  $H$ -space product  $m_{\mathbb{C}P^\infty} : \mathbb{C}P^\infty \times \mathbb{C}P^\infty \rightarrow \mathbb{C}P^\infty$  induces a coproduct  $m_{\mathbb{C}P^\infty}^*$  on  $BP^*(\mathbb{C}P^\infty)$  with  $m_{\mathbb{C}P^\infty}^*(x^{\mathbb{C}P^\infty}) = \sum_{i,j \geq 0} a_{ij} (x^{\mathbb{C}P^\infty})^i \otimes (x^{\mathbb{C}P^\infty})^j$  where  $a_{ij} \in BP^{-2(i+j-1)} = BP_{2(i+j-1)}$ .
7.  $F(y, z) = y +_{F_{BP}} z = y +_F z = \sum_{i,j \geq 0} a_{ij} y^i z^j$  is a commutative associative formal group law over  $BP^*$ .

Let  $\beta(s) = \sum_{i \geq 0} \beta_i s^i$  where  $s$  is a formal indeterminate.

**Proposition 2.4** (In [RW77] Theorem 3.4 on Page 255) *In the power series ring  $BP_*(\mathbb{C}P^\infty)[[s, t]]$ , we have*

$$\beta(s) \beta(t) = \beta(s +_F t)$$

where the product is that induced by the  $H$ -space structure of  $\mathbb{C}P^\infty$ . Moreover, for a natural number  $n$ ,

$$\beta(s)^n = \beta([n]_F(s)),$$

where  $[n]_F(s) = s +_F s +_F \cdots +_F s$  for  $n > 0$ .

Consider  $x \in BP^2(\mathbb{C}P^\infty) = [\mathbb{C}P^\infty, \underline{BP}_2]$  as a map

$$\mathbb{C}P^\infty \xrightarrow{x} \underline{BP}_2.$$

This map induces a map  $x_* : BP_*(\mathbb{C}P^\infty) \rightarrow BP_*(\underline{BP}_2)$ .

**Definition 2.5** *We define*

$$b_i = x_*(\beta_i) \in BP_{2i}(\underline{BP}_2).$$

As with  $\beta(s)$ , we have

$$x_*(\beta(s)) = x_*\left(\sum_{i \geq 0} \beta_i s^i\right) = \sum_{i \geq 0} x_*(\beta_i) s^i = \sum_{i \geq 0} b_i s^i.$$

Therefore, we define  $b(s) = \sum_{i \geq 0} b_i s^i \in BP_*(\underline{BP}_2)[[s]]$ , that is,  $b(s) = x_*(\beta(s))$ .

The  $H$ -space product  $m : \underline{BP}_2 \times \underline{BP}_2 \rightarrow \underline{BP}_2$  induces a coproduct  $m^*$  on  $BP^*(\underline{BP})$  with  $m^*(x) = \sum_{i,j \geq 0} a_{ij} x^i \otimes x^j$  where  $a_{ij} \in BP^{-2(i+j-1)} = BP_{2(i+j-1)}$ .

According to [RW80],  $BP_*(\underline{BP}_*)$  is a Hopf ring, where the maps of coalgebras

$$* : BP_*(\underline{BP}_k) \otimes BP_*(\underline{BP}_k) \cong BP_*(\underline{BP}_k \times \underline{BP}_k) \rightarrow BP_*(\underline{BP}_k)$$

are induced by the maps  $\underline{BP}_k \times \underline{BP}_k \rightarrow \underline{BP}_k$  that represent addition in cohomology and the maps of coalgebras

$$\circ : BP_*(\underline{BP}_k) \otimes BP_*(\underline{BP}_m) \cong BP_*(\underline{BP}_k \times \underline{BP}_m) \rightarrow BP_*(\underline{BP}_{k+m})$$

are induced by the maps  $\underline{BP}_k \times \underline{BP}_m \rightarrow \underline{BP}_{k+m}$  that represent multiplication in cohomology.

**Definition 2.6** *In  $BP_*(\underline{BP}_*)[[s, t]]$ , we define*

$$b(s) +_{[F]} b(t) = b(s) +_{[F]BP} b(t) = \sum_{i,j \geq 0}^* [a_{ij}] \circ b(s)^{\circ i} \circ b(t)^{\circ j}.$$

**Proposition 2.7** (In [RW77] Theorem 3.8 on Page 256) *In  $BP_*(\underline{BP}_*)[[s, t]]$ , we have relations:*

1.  $b(s +_F t) = b(s) +_{[F]} b(t)$ ,
2.  $b([2]_F(s)) = [2]_{[F]}(b(s))$ .

We can rewrite the relation  $b(s +_F t) = b(s) +_{[F]} b(t)$  as

$$b\left(\sum_{i,j \geq 0} a_{ij} s^i t^j\right) = \sum_{i,j \geq 0}^* [a_{ij}] \circ b(s)^{\circ i} \circ b(t)^{\circ j},$$

that is,

$$\sum_{n \geq 0} b_n \left( \sum_{i,j \geq 0} a_{ij} s^i t^j \right)^n = \underset{i,j \geq 0}{*} [a_{ij}] \circ \left( \sum_{k \geq 0} b_k s^k \right)^{\circ i} \circ \left( \sum_{l \geq 0} b_l s^l \right)^{\circ j}.$$

**Proposition 2.8** (In [RW77] Theorem 3.11(b) on Page 257) *In BP for the prime 2, we have*

$$[2]_F(t) = \sum_{n > 0}^{F_{BP}} v_n t^{2^n} \text{ mod } (2).$$

In BP with prime 2, we have, see [RW77],

$$[2]_F(t) = 2t - v_1 t^2 + 2v_1^2 t^3 - (7v_2 + 8v_1^3) t^4 + (30v_2 v_1 + 26v_1^4) t^5 - (111v_2 v_1^2 + 84v_1^5) t^6 + \dots$$

**Remark 2.9** From Lemma 1.12 in [RW77],

1.  $a_{10} = a_{01} = 1$  and  $a_{i0} = a_{0i} = 0$  if  $i \neq 1$ ,
2.  $b_0 = [0_2]$ ,
3.  $b_0 \circ b_i = 0$  for all  $i > 0$ ,
4.  $b_0 \circ [a] = [0_{2-2i}]$  where  $a \in BP^{-2i}$ ,
5.  $2b_1 = [2] \circ b_1$ ,
6.  $b_1^{*2} = 2b_2 - v_1 b_1 + [v_1] \circ b_1^{\circ 2}$  in  $BP_2$  ( $\underline{BP}_2$ ) from page 261.

## 2.2 Wilson's Splitting Theorem

We study the commutative diagram of  $H$ -spaces and canonical  $H$ -maps

$$\begin{array}{ccc} \underline{BP}_6 & \xrightarrow{v_1} & \underline{BP}_4 \\ \downarrow & & \downarrow \\ K(\mathbb{Z}_{(2)}, 3) & \xrightarrow{f} & \underline{BP}\langle 1 \rangle_6 \xrightarrow{v_1} \underline{BP}\langle 1 \rangle_4 \end{array} \quad (1)$$

in which the bottom row is a fibration.

By Wilson [Wil73],  $\underline{BP}\langle 1 \rangle_4$  splits off  $\underline{BP}_4$  as an  $H$ -space.  $\underline{BP}\langle 1 \rangle_6$  splits off  $\underline{BP}_6$ , but not as an  $H$ -space. Both maps  $v_1$  represent multiplication by  $v_1$  in cohomology.

(The fibre of an  $H$ -map is automatically an  $H$ -space, and by uniqueness, the resulting  $H$ -space structure on  $K(\mathbb{Z}_{(2)}, 3)$  must be the standard one, up to homotopy.)

We apply homology theories  $E_*(-)$  and cohomology theories  $E^*(-)$  to this diagram, for various  $E = BP$  or  $E(1, 2)$ . Note that since all the  $E$  we consider are 2-local, we may replace  $E^*(K(\mathbb{Z}_{(2)}, 3))$  by  $E^*(K(\mathbb{Z}, 3))$ .  $BP_*(\underline{BP}_k)$  was computed by Ravenel-Wilson [RW77] and is a free  $BP_*$ -module. Since each  $E$  we use comes with a multiplicative map  $BP \rightarrow E$  that induces a surjection  $BP_* \rightarrow E_*$ , comparison of the Atiyah-Hirzebruch spectral sequences shows that  $E_*(\underline{BP}_k)$  and  $E_*(\underline{BP}\langle 1 \rangle_k)$  are free  $E_*$ -modules for  $k = 4, 6$ . More precisely,

$$E_* \otimes_{BP_*} BP_*(\underline{BP}_k) \cong E_*(\underline{BP}_k).$$

In view of the above splittings, the same holds for  $\underline{BP}\langle 1 \rangle_k$ . By dualizing, we have

$$E^* \widehat{\otimes}_{BP^*} BP^*(\underline{BP}_k) \cong E^*(\underline{BP}_k)$$

in cohomology.

Ravenel-Wilson-Yagita [RWY] established the short exact sequence of bicommulative Hopf algebras

$$E^* \longrightarrow E^*(\underline{BP}\langle 1 \rangle_4) \xrightarrow{v_1^*} E^*(\underline{BP}\langle 1 \rangle_6) \xrightarrow{f^*} E^*(K(\mathbb{Z}, 3)) \longrightarrow E^*$$

for various  $E$ . We wish to use this to compute  $E(1, 2)^*(K(\mathbb{Z}, 3))$ .

### 2.3 The Formal Group Law for $E(1, 2)$

Consider the spectrum  $E(1, 2)$  associated with the prime 2 where the coefficient ring is  $E(1, 2)^* = \mathbb{Z}/2[v_1, v_2, v_2^{-1}]$ . We study its formal group law  $F(y, z)$ .

For convenience, we define  $A_1 = v_1$  and  $A_2 = v_2$ . Also, for  $k \geq 3$ , we define

$$A_k = \sum_{\substack{m, n \geq 1 \\ 2m+2^n=2k}} a_{mn} v_1^m v_2^n.$$

**Lemma 2.10**  $v_1 t^2 +_F v_2 t^4 = \sum_{k \geq 1} A_k t^{2k}$ .

**Proof.** By the formal group law, we have

$$v_1 t^2 +_F v_2 t^4 = v_1 t^2 + v_2 t^4 + \sum_{m,n \geq 1} a_{mn} v_1^m v_2^n t^{2m+2n}.$$

Take the coefficient of  $t^{2k}$ . ■

A lower degree calculation (2.11) is in the Appendix.

We note that  $(a+b)^2 \equiv a^2 + b^2 \pmod{2}$ . Thus, we have

$$(v_1 t^2 +_F v_2 t^4)^{2^q} = \sum_{k \geq 1} A_k^{2^q} t^{2^{q+1}k}.$$

In general, we define  $C_{q,k}$  as the coefficient of  $t^{2k}$  in the equation

$$(v_1 t^2 +_F v_2 t^4)^q = \sum_{k \geq 1} C_{q,k} t^{2k}.$$

It's easy to see that

$$C_{q,k} = 0, \text{ if } k < q$$

and

$$C_{2^q,k} = A_{2^{-q}k}^{2^q}.$$

for all  $q \geq 1$  and  $k \geq 2^q$ , interpreted as 0 if  $k$  is not divisible by  $2^q$ .

## 2.4 Appendix: Low Degree Examples

We collect some precise low degree calculations here for reference.

**Example 2.11** *The first few terms of*

$$\begin{aligned} & v_1 t^2 +_F v_2 t^4 \\ = & v_1 t^2 + v_2 t^4 + a_{11} v_1 v_2 t^6 + a_{21} v_1^2 v_2 t^8 + (a_{31} v_1^3 v_2 + a_{12} v_1 v_2^2) t^{10} \\ & + (a_{41} v_1^4 v_2 + a_{22} v_1^2 v_2^2) t^{12} + (a_{51} v_1^5 v_2 + a_{32} v_1^3 v_2^2 + a_{13} v_1 v_2^3) t^{14} \\ & + (a_{61} v_1^6 v_2 + a_{42} v_1^4 v_2^2 + a_{23} v_1^2 v_2^3) t^{18} + \dots \\ = & A_1 t^2 + A_2 t^4 + A_3 t^6 + A_4 t^8 + A_5 t^{10} + A_6 t^{12} + A_7 t^{14} + A_8 t^{16} + \dots \end{aligned}$$

**Example 2.12** *The first few terms of*

$$\begin{aligned}
& (v_1 t^2 +_F v_2 t^4)^2 \\
&= \left( \sum_{k \geq 1} A_k t^{2k} \right)^2 \\
&= (A_1 t^2 + A_2 t^4 + A_3 t^6 + A_4 t^8 + \dots) (A_1 t^2 + A_2 t^4 + A_3 t^6 + A_4 t^8 + \dots) \\
&= A_1 A_1 t^4 + (A_1 A_2 + A_2 A_1) t^6 + (A_1 A_3 + A_2 A_2 + A_3 A_1) t^8 \\
&\quad + (A_1 A_4 + A_2 A_3 + A_3 A_2 + A_4 A_1) t^{10} + \dots \\
&= A_1^2 t^4 + 2A_1 A_2 t^6 + (A_2^2 + 2A_1 A_3) t^8 + 2(A_1 A_4 + A_2 A_3) t^{10} + \dots
\end{aligned}$$

or

$$\begin{aligned}
& (v_1 t^2 +_F v_2 t^4)^2 \\
&= v_1^2 t^4 + 2v_1 v_2 t^6 + (v_2^2 + 2a_{11} v_1^2 v_2) t^8 + 2(a_{21} v_1^3 v_2 + a_{11} v_1 v_2^2) t^{10} \\
&\quad + (a_{11}^2 v_1^2 v_2^2 + 2(a_{31} v_1^4 v_2 + a_{12} v_1^2 v_2^2 + a_{21} v_1^2 v_2^2)) t^{12} \\
&\quad + 2(a_{12} v_1 v_2^3 + a_{41} v_1^5 v_2 + a_{22} v_1^3 v_2^2 + a_{31} v_1^3 v_2^2 + a_{11} a_{21} v_1^3 v_2^2) t^{14} + (a_{21}^2 v_1^4 v_2^2) t^{16} \\
&\quad + 2(a_{51} v_1^6 v_2 + a_{13} v_1^2 v_2^3 + a_{22} v_1^2 v_2^3 + a_{32} v_1^4 v_2^2 + a_{41} v_1^4 v_2^2 + a_{11} a_{12} v_1^2 v_2^3 + a_{11} a_{31} v_1^4 v_2^2) t^{16} \\
&\quad + 2(a_{32} v_1^3 v_2^3 + a_{51} v_1^5 v_2^2 + a_{13} v_1 v_2^4 + a_{61} v_1^7 v_2 + a_{42} v_1^5 v_2^2 + a_{23} v_1^3 v_2^3) t^{18} \\
&\quad + 2(a_{11} a_{22} v_1^3 v_2^3 + a_{12} a_{21} v_1^3 v_2^3 + a_{11} a_{41} v_1^5 v_2^2 + a_{21} a_{31} v_1^5 v_2^2) t^{18} + \dots
\end{aligned}$$

**Example 2.13** *For  $p = 2$ , the first few terms of*

$$\begin{aligned}
& (v_1 t^2 +_F v_2 t^4)^{2^2} \\
&= \left( \sum_{k \geq 1} A_k t^{2k} \right)^{2^2} \\
&= (A_1 t^2 + A_2 t^4 + A_3 t^6 + \dots)^4 \\
&= A_1^4 t^8 + 2^2 A_1^3 A_2 t^{10} + 2(2A_1^3 A_3 + 3A_1^2 A_2^2) t^{12} \\
&\quad + (A_2^4 + 6A_1^2 A_3^2 + 12A_1 A_2^2 A_3 + 12A_1^2 A_2 A_4 + 4A_1^3 A_5) t^{16} + \dots
\end{aligned}$$

### 3 $BP$ Homology and Cohomology of $\underline{BP}\langle 1\rangle_4$ and $\underline{BP}\langle 1\rangle_6$

We focus on the prime 2. Let  $E$  be a 2-local cohomology theory. We borrow the notations and results from [RW77]:

**Definition 3.1**  $[v_i]$  is defined in  $E_0(\underline{BP}_{-2(2^i-1)})$  using  $v_i \in \pi_{2(2^i-1)}(BP)$ .

**Definition 3.2**  $\Delta_k = (0, \dots, 0, 1, 0, \dots)$  where 1 is in the  $k$ -th position.

**Definition 3.3**  $b_{(m)} = b_{2^m} \in E_{2^{m+1}}(\underline{BP}_2)$ .

**Definition 3.4** In the ring of indecomposables  $QE_*(\underline{BP}_*)$ , define

$$[v^I] b^J = [v_1^{i_1} v_2^{i_2} \dots] \circ b_{(0)}^{\circ j_0} \circ b_{(1)}^{\circ j_1} \circ \dots$$

where  $I = (i_1, i_2, \dots)$  and  $J = (j_0, j_1, \dots)$  are sequences of non-negative integers almost all zero. Note that Proposition 2.7 implies that the other  $b_n$  are redundant.

**Definition 3.5** For  $J = (j_0, j_1, \dots)$ , we define the length of  $J$  as  $\ell(J) = j_0 + j_1 + \dots$ .

**Definition 3.6** We call  $[v^I] b^J$  **allowable** if

$$J = 2\Delta_{k_1} + 2^2\Delta_{k_2} + \dots + 2^n\Delta_{k_n} + J',$$

where  $k_1 \leq k_2 \leq \dots \leq k_n$  and  $J'$  is non-negative, implies  $i_n = 0$ .

**Proposition 3.7** The allowable  $[v^I] b^J$  ( $J \neq 0$ ) form a basis of the free  $BP_*$ -module  $QBP_*\left(\underline{BP}'_*\right)$ , where  $\underline{BP}'_*$  denotes the base components of  $\underline{BP}_*$ . ( $\underline{BP}'_k = \underline{BP}_k$  for  $k > 0$ .)

**Proposition 3.8** The  $[v^I] b^J \circ b_{(0)}$  with  $[v^I] b^J$  allowable ( $J$  possibly zero) form a basis of the submodule of primitives  $PBP_*\left(\underline{BP}'_*\right)$ .

**Remark 3.9** A basis of  $QBP_{2^m}(\underline{BP}_{2^n})$ ,  $n > 0$  is given by all allowable  $[v^I] b^J$  with  $m = \sum_{k \geq 0} j_k 2^k$  and  $n = \sum_{k \geq 0} j_k - \sum_{k > 0} i_k (2^k - 1)$ .

We need to compute  $v_1^* : E^* \left( \underline{BP} \langle 1 \rangle_4 \right) \longrightarrow E^* \left( \underline{BP} \langle 1 \rangle_6 \right)$ . These are power series rings. We compute on the indecomposables,  $Qv_1^* : QE^* \left( \underline{BP} \langle 1 \rangle_4 \right) \longrightarrow QE^* \left( \underline{BP} \langle 1 \rangle_6 \right)$ , which is dual to  $Pv_{1*} : PE_* \left( \underline{BP} \langle 1 \rangle_6 \right) \longrightarrow PE_* \left( \underline{BP} \langle 1 \rangle_4 \right)$ . The latter is simply multiplication by  $[v_1]$ .

The double suspension factors as

$$b_{(0)} \circ - : E_* \left( \underline{BP} \langle 1 \rangle_4 \right) \longrightarrow QE_* \left( \underline{BP} \langle 1 \rangle_4 \right) \xrightarrow{\cong} PE_* \left( \underline{BP} \langle 1 \rangle_6 \right) \hookrightarrow E_* \left( \underline{BP} \langle 1 \rangle_6 \right)$$

and similarly for  $PE_* \left( \underline{BP} \langle 1 \rangle_4 \right)$ . We therefore compute the homomorphism of free  $E_*$ -modules

$$v_{1*} : QE_* \left( \underline{BP} \langle 1 \rangle_4 \right) \longrightarrow QE_* \left( \underline{BP} \langle 1 \rangle_2 \right).$$

We temporarily omit  $\circ$  here, as it is the only multiplication in  $QE_* \left( \underline{BP} \langle 1 \rangle_* \right)$ .

The Ravenel-Wilson basis is not the most useful here.

**Theorem 3.10** *The elements  $b_i$  for  $i > 0$  form a basis of  $QE_* \left( \underline{BP} \langle 1 \rangle_2 \right)$  as a free  $E_*$ -module.*

To show that a set of elements is a basis, it is sufficient to work mod  $I_\infty = (2, v_1, v_2, \dots)$  and check that we have a basis over  $\mathbb{F}_p$ , according to the Nakayama Lemma. We introduce the Bendersky generators  $h_i$  of  $QE_* \left( \underline{BP}_* \right)$  (see [B82]). They may be defined by

$$b(x) = h_0 x +_{[F]} h_1 x^2 +_{[F]} h_2 x^4 +_{[F]} \dots \quad (2)$$

where  $[F]$  denotes the formal group law using the right  $BP_*$ -action by the elements  $[v_i]$ . If we replace each  $b_{(i)}$  by  $h_i$  in the Ravenel-Wilson basis of  $QE_* \left( \underline{BP}_* \right)$ , we obtain another basis (since  $h_i \equiv b_{(i)} \pmod{I_\infty}$ , see below.) When we map to  $\underline{BP} \langle 1 \rangle_2$ , the resulting basis of  $E_* \left( \underline{BP} \langle 1 \rangle_2 \right)$  consists of the elements

$$[v_1^k] h_{i_0} h_{i_1} h_{i_2} \cdots h_{i_k}$$

where  $k \geq 0$  and  $i_0 < i_1 < i_2 < \cdots < i_k$ .

The theorem follows from the following lemma.

**Lemma 3.11** *Given  $n > 0$ , write  $n$  in binary form as  $n = 2^{i_0} + 2^{i_1} + \cdots + 2^{i_k}$ , with  $i_0 < i_1 < \cdots < i_k$ . Then in  $QE_*(\underline{BP}_2)$ ,*

$$b_n \equiv [v_1^k] h_{i_0} h_{i_1} h_{i_2} \cdots h_{i_k} \text{ mod } (I_\infty + ([v_2], [v_3], \cdots)).$$

**Proof.** The main relation (see Proposition 2.7) reduces to

$$0 \equiv b([2](x)) \equiv [v_1] b(x)^2 \equiv \sum_i [v_1] b_i^2 x^{2i}.$$

The coefficient of  $x^{2^{i+1}}$  yields  $[v_1] b_{(i)}^2 \equiv 0$ .

We deduce by induction on  $i$  that  $[v_1] h_i^2 \equiv 0$ , starting from  $b_{(0)} = h_0$ . When we expand (2) and take the coefficients of  $x^{2^i}$ , we get  $b_{(i)} = h_i + \text{terms with a factor } [v_1] h_j^2$  for some  $j < i$ . Hence  $b_{(i)} \equiv h_i$  and  $[v_1] h_i^2 \equiv 0$ .

If  $[v_1] x^2 \equiv 0$  and  $[v_1] y^2 \equiv 0$ , the right formal group law takes the form  $x +_{[F]} y \equiv x + y + [v_1] xy$ . Then the coefficient of  $x^n$  in (2) yields  $b_n$  on the left, and the only surviving term on the right is  $[v_1^k] h_{i_0} h_{i_1} h_{i_2} \cdots h_{i_k}$ . ■

**Lemma 3.12** *In  $QE_*(\underline{BP}\langle 1 \rangle_2)$ , each  $b_n$  can be expressed as a polynomial in the elements  $b_{(i)}$  and  $[v_1]$ , with no  $v_1$  or  $v_2$ .*

**Proof.** The coefficient of  $x^n$  in (2) expresses  $b_n$  as a polynomial in the  $h_i$  and  $[v_1]$ . If  $n = 2^i$ , we can use  $b_{(i)} = h_i + \cdots$  to express  $h_i$  as a polynomial in  $b_{(i)}$ , the  $h_j$  for  $j < i$ , and  $[v_1]$ , and hence inductively as a polynomial in the  $b_{(j)}$  and  $[v_1]$ . ■

**Remark 3.13** If  $n$  is not a power of 2, this polynomial is divisible by  $[v_1]$ .

Next, we consider  $QE_*(\underline{BP}\langle 1 \rangle_4)$ .

**Lemma 3.14** *If  $n$  is not a power of 2, there is an element  $\hat{b}_n \in QE_*(\underline{BP}\langle 1 \rangle_4)$  such that  $[v_1] \hat{b}_n = b_n$  in  $QE_*(\underline{BP}\langle 1 \rangle_2)$ .*

**Proof.** When we expand (2) and take the coefficient of  $x^n$  to find  $b_n$ , every term on the right (working mod  $[v_2], [v_3], \cdots$ ) contains a factor  $[v_1]$ . ■

In particular, by Lemma 3.11,

$$\hat{b}_n \equiv [v_1^{k-1}] h_{i_0} h_{i_1} h_{i_2} \cdots h_{i_k} \text{ mod } I_\infty.$$

The only missing elements from the modified Ravenel-Wilson basis of  $QE_* \left( \underline{BP} \langle 1 \rangle_4 \right)$  are the elements  $h_i^2 \equiv b_{(i)}^2$  for  $i \geq 0$ . We therefore take as a basis of  $QE_* \left( \underline{BP} \langle 1 \rangle_4 \right)$ , the elements  $\hat{b}_n$  for  $n$  not a power of 2 and the elements  $b_{(i)}^2$ . We now compute  $v_{1*}$ . The main relation, mod 2, is

$$\sum_q b_q ([2](x))^q = b([2](x)) \equiv [v_1] b(x)^2 \equiv \sum_i [v_1] b_i^2 x^{2i}.$$

Recall that  $([2](x))^q = \sum_{k \geq 1} C_{q,k} x^{2k}$ . Take coefficients of  $x^{2^{i+1}}$ . Then

$$\begin{cases} v_{1*} b_{(i)}^2 = [v_1] b_{(i)}^2 = \sum_{q \leq 2^i} C_{q, 2^i} b_q, & \text{for } i \geq 0; \\ v_{1*} \hat{b}_n = [v_1] \hat{b}_n = b_n, & \text{for } n \text{ not a power of 2} \end{cases} \quad (3)$$

gives  $v_{1*}$  precisely in terms of our bases.

We summarize the relationship between  $E_*$ ,  $QE_*$  and  $PE_*$  in a diagram:

$$\begin{array}{ccc} E_* \left( \underline{BP} \langle 1 \rangle_4 \right) & \xrightarrow{v_{1*}} & E_* \left( \underline{BP} \langle 1 \rangle_2 \right) \\ \downarrow & & \downarrow \\ QE_* \left( \underline{BP} \langle 1 \rangle_4 \right) & \xrightarrow{Qv_{1*}} & QE_* \left( \underline{BP} \langle 1 \rangle_2 \right) \\ \cong \downarrow - \circ b_{(0)} & & \cong \downarrow \\ PE_* \left( \underline{BP} \langle 1 \rangle_6 \right) & \xrightarrow{Pv_{1*}} & PE_* \left( \underline{BP} \langle 1 \rangle_4 \right) \\ \cap & & \cap \\ E_* \left( \underline{BP} \langle 1 \rangle_6 \right) & \xrightarrow{v_{1*}} & E_* \left( \underline{BP} \langle 1 \rangle_4 \right) & \xrightarrow{\langle x_{(i)}, - \rangle} & E_* \\ \downarrow & & \downarrow & & \\ QE_* \left( \underline{BP} \langle 1 \rangle_6 \right) & \xrightarrow{Qv_{1*}} & QE_* \left( \underline{BP} \langle 1 \rangle_4 \right) & & \end{array} \quad (4)$$

### 3.1 Generators of Cohomology

We return to the primitives. Our basis of  $PE_* \left( \underline{BP} \langle 1 \rangle_4 \right) = PM_4 \oplus PR_4$  consists of two parts:

1. Elements  $b_{(0)} \circ b_n$  for  $n$  not a power of 2, which span the submodule  $PM_4$ .
2. Elements  $b_{(0)} \circ b_{(i)}$  for  $i \geq 0$ , which span the submodule  $PR_4$ .

Our basis of  $PE_* \left( \underline{BP} \langle 1 \rangle_6 \right) = PM_6 \oplus PR_6$  consists of two parts:

1. Elements  $b_{(0)} \circ \hat{b}_n$  for  $n$  not a power of 2, which span the submodule  $PM_6$ .
2. Elements  $b_{(0)} \circ b_{(i)}^{\circ 2}$  for  $i \geq 0$ , which span the submodule  $PR_6$ .

In terms of these bases,  $v_{1*}$  is given by

$$\begin{cases} v_{1*} \left( b_{(0)} \circ \hat{b}_n \right) = [v_1] \circ b_{(0)} \circ \hat{b}_n = b_{(0)} \circ b_n, & \text{for } n \text{ not a power of 2} \\ v_{1*} \left( b_{(0)} \circ b_{(i)}^{\circ 2} \right) = [v_1] \circ b_{(0)} \circ b_{(i)}^{\circ 2} = \sum_{q \leq 2^i} C_{q,2^i} b_{(0)} \circ b_q. \end{cases} \quad (5)$$

Note that  $v_{1*} \left( b_{(0)} \circ b_{(i)}^{\circ 2} \right)$  contains terms in  $PM_4$ .

Now, we can consider the duals  $QE^* \left( \underline{BP} \langle 1 \rangle_4 \right)$  and  $QE^* \left( \underline{BP} \langle 1 \rangle_6 \right)$ , the indecomposables of the algebras  $E^* \left( \underline{BP} \langle 1 \rangle_4 \right)$  and  $E^* \left( \underline{BP} \langle 1 \rangle_6 \right)$ .

**Proposition 3.15** (In [RWY] Page 6) *Since, for  $k \leq 2(2^q + 2^{q-1} + \dots + 2 + 1)$ , the spaces  $\underline{BP} \langle q \rangle_k$  are all torsion-free for ordinary homology, we know that they are  $BP_*$ -free and the Brown-Peterson cohomology is just the  $BP^*$ -dual. Likewise, the induced cohomology homomorphisms are just the dual homomorphisms.*

**Proposition 3.16** (In [RWY] Section 8.5 at Page 39) *Both  $BP^* \left( \underline{BP} \langle 1 \rangle_6 \right)$  and  $BP^* \left( \underline{BP} \langle 1 \rangle_4 \right)$  are power series rings on generators dual to the primitives in the  $BP$ -homology.*

Dualizing to cohomology, we write  $E^* \left( \underline{BP} \langle 1 \rangle_4 \right) = A \hat{\otimes} M$ , where  $A$  and  $M$  are power series rings generated by the duals of  $PR_4$  and  $PM_4$ , respectively, and  $E^* \left( \underline{BP} \langle 1 \rangle_6 \right) = B \hat{\otimes} N$ , where  $B$  and  $N$  are power series rings generated by the duals of  $PR_6$  and  $PM_6$ , respectively. Since  $v_{1*} : PM_6 \cong PM_4$  is an isomorphism, so is  $Qv_1^* : QM \rightarrow QN$  (considered as quotient modules). So

$$\text{coker} [Qv_1^* : QA \oplus QM \rightarrow QB \oplus QN] \cong \text{coker} [Qv_1^* : QA \rightarrow QB].$$

We therefore limit attention to  $A$  and  $B$ . Define generators  $x_{(i)} \in A$  dual to  $b_{(0)} \circ b_{(i)}$  and  $y_{(i)} \in B$  dual to  $b_{(0)} \circ b_{(i)}^{\circ 2}$ , so that

$$A = E^* \left[ [x_{(i)} : i \geq 0] \right]$$

and

$$B = E^* \left[ [y_{(i)} : i \geq 0] \right].$$

It is to be understood that all such rings allow any infinite linear combination of monomials with coefficients in  $E^*$  of appropriate degrees. Note that the degree of  $x_{(i)}$  is  $2(1 + 2^i)$  and the degree of  $y_{(i)}$  is  $2(1 + 2^{i+1})$ . Then dualizing (5), in  $Qv_1^* : QE^* \left( \underline{BP} \langle 1 \rangle_4 \right) \longrightarrow QE^* \left( \underline{BP} \langle 1 \rangle_6 \right)$ , we have

$$v_1^* (x_{(i)}) = \sum_{j \geq i} C_{2^i, 2^j} y_{(j)}.$$

Recall that  $C_{2^i, 2^j} = A_{2^{j-i}}^{2^i} = A_{(j-i)}^{2^i}$ , where we write  $A_{(i)} = A_{2^i}$ , and that  $A_{(0)} = A_1 = v_1$  and  $A_{(1)} = A_2 = v_2$ . Then, for  $i \geq 0$ ,  $Qv_1^* : QA \longrightarrow QB$  is given by

$$v_1^* (x_{(i)}) = v_1^{2^i} y_{(i)} + v_2^{2^i} y_{(i+1)} + \sum_{k \geq 2} A_{(k)}^{2^i} y_{(i+k)}. \quad (6)$$

Also,

$$v_1^* (u_n) = w_n + \sum_j C_{n, 2^j} y_{(j)},$$

where  $u_n$  is dual to  $b_{(0)} \circ b_n$  and  $w_n$  is dual to  $b_{(0)} \circ \hat{b}_n$ .

### 3.2 The Module $QE^* (K(\mathbb{Z}, 3))$

We consider the indecomposables of the algebra  $E^* (K(\mathbb{Z}, 3))$ , where  $E = E(1, 2)$ . Put  $z_{(i)} = f^* (y_{(i)}) \in Q = QE^* (K(\mathbb{Z}, 3))$ ; these elements generate the  $E^*$ -module topologically, with respect to the  $z$ -filtration in which  $F^n = F^n Q$  consists of all elements  $\sum_{k \geq n} d_k z_{(k)}$ , with  $d_k \in E^*$ . From (6), we have the relations

$$v_1^{2^i} z_{(i)} + v_2^{2^i} z_{(i+1)} + \sum_{k \geq 2} A_{(k)}^{2^i} z_{(i+k)} = 0$$

for  $i \geq 0$ . Note that for  $k \geq 2$ ,  $A_{(k)}$  is divisible by  $v_1 v_2$ ; write  $A_{(k)} = v_1 v_2 A'_{(k)}$ .

We also have the  $v_1$ -filtration, defined by the submodules  $v_1^k Q$ . Since  $v_2$  is invertible, we may rewrite the above relation as

$$z_{(i+1)} = v_1^{2^i} \left( v_2^{-2^i} z_{(i)} + \sum_{k \geq 2} (A'_{(k)})^{2^i} z_{(i+k)} \right).$$

**Lemma 3.17**  $Q/v_1^{2^n}Q$  is a free  $E^*/(v_1^{2^n})$ -module generated by  $z_{(0)}$ .

**Proof.** It is easy to see that  $z_{(0)}$  generates  $Q/v_1^{2^n}Q$ . We have  $Q = E^*z_{(0)} + v_1Q$ . Multiply by  $v_1^i$ ; then

$$E^*z_{(0)} + v_1^iQ = E^*z_{(0)} + v_1^iE^*z_{(0)} + v_1^{i+1}Q = E^*z_{(0)} + v_1^{i+1}Q.$$

By induction,

$$Q = E^*z_{(0)} + v_1Q = E^*z_{(0)} + v_1^{2^n}Q.$$

To see that there are no relations, let  $F$  be the free  $R$ -module on generators  $y_{(0)}, y_{(1)}, \dots, y_{(n)}$ , considered as a quotient module of  $QE^*(\underline{BP}\langle 1 \rangle_6)$  via  $q : QE^*(\underline{BP}\langle 1 \rangle_6) \rightarrow F$ , where for convenience we write  $R = E^*/(v_1^{2^n})$ . We note that  $Q/v_1^{2^n}Q$  is a quotient of  $F$ . Suppose  $cz_{(0)} = 0$  in  $Q/v_1^{2^n}Q$ , where  $c \in R$ . Since  $qv_1^*(x_{(i)}) = 0$  for  $i \geq n$ , this lifts to a relation

$$cy_{(0)} = \sum_{i=0}^{n-1} c_i qv_1^*(x_{(i)}) = \sum_{i=0}^{n-1} c_i \left( v_1^{2^i} y_{(i)} + v_2^{2^i} y_{(i+1)} + \sum_{k=2}^{n-i} v_1^{2^i} v_2^{2^i} (A'_{(k)})^{2^i} y_{(i+k)} \right)$$

in  $F$ , for some coefficients  $c_i \in R$ . The coefficient of  $y_{(0)}$  gives  $c = v_1c_0$ . The coefficient of  $y_{(1)}$  gives  $0 = v_1^2c_1 + v_2c_0$ , or  $c_0 = v_1^2v_2^{-1}c_1$ . We show by induction on  $i$  that  $c_{i-1} = v_1^{2^i}v_2^{-2^{i-1}}(1 + a_i)c_i$  for  $0 < i \leq n-1$ , for some  $a_i \in (v_1) \subset R$ . Assume this holds for  $i < m$ . The coefficient of  $y_{(m)}$ , where  $0 < m \leq n-1$ , in the above relation gives

$$\begin{aligned} 0 &= v_1^{2^m}c_m + v_2^{2^{m-1}}c_{m-1} + \sum_{k=2}^m v_1^{2^{m-k}}v_2^{2^{m-k}}(A'_{(k)})^{2^{m-k}}c_{m-k} \\ &= v_1^{2^m}c_m + v_2^{2^{m-1}}(1 + v_1A)c_{m-1} \end{aligned}$$

where by the induction hypothesis we express each  $c_{m-k}$  in terms of  $c_{m-1}$  and  $A \in R$ . Now  $1 + v_1A$  is invertible in  $R$  by  $(1 + v_1A)^{-1} = 1 + a_m$ , where  $a_m = \sum_{i=1}^{2^n-1} v_1^i A^i$ , so that  $c_{m-1} = v_1^{2^m}v_2^{-2^{m-1}}(1 + a_m)c_m$ .

The same argument applies when  $m = n$ , except that there is no  $c_n$ , to yield  $c_{n-1} = 0$ . Then  $c_i = 0$  for all  $i$  and  $c = v_1c_0 = 0$ . ■

**Theorem 3.18** The  $z$ -filtration and the  $v_1$ -filtration define the same topology on  $Q = QE^*(K(\mathbb{Z}, 3))$ .  $Q$  is the free  $\mathbb{F}_2[v_2, v_2^{-1}][[v_1]]$ -module generated by  $z_{(0)}$ .

**Proof.** We have already seen that  $z_{(i+1)} \in v_1^{2^i} Q$  for all  $i \geq 0$  so that  $F^n \subset v_1^{2^{n-1}} Q$ . Conversely,  $v_1^{2^n-1} Q \subset F^n$ . This follows from  $v_1^{2^n-2^i} z_{(i)} \in F^n$  for  $0 \leq i \leq n$ , by downward induction on  $i$ , starting from  $i = n$ . For the induction step, we multiply the above relation by  $v_1^{2^n-2^{i+1}}$ .

We know the  $z$ -filtration is complete, being a quotient of the complete filtered  $E^*$ -module  $QE_* \left( \underline{BP} \langle 1 \rangle_6 \right)$ . By the above, the  $v_1$ -filtration is also complete, and

$$Q = \lim_n Q / v_1^{2^n} Q \cong \lim_n \mathbb{F}_2 [v_2, v_2^{-1}] [v_1] / (v_1^{2^n}) = \mathbb{F}_2 [v_2, v_2^{-1}] [[v_1]].$$

■

We would like to extend this result to the whole algebra  $E^*(K(\mathbb{Z}, 3))$ .

### 3.3 Bases of Homology and Cohomology

We need to extend bases of  $PE_* \left( \underline{BP} \langle 1 \rangle_6 \right)$  and  $PE_* \left( \underline{BP} \langle 1 \rangle_4 \right)$  to  $E_* \left( \underline{BP} \langle 1 \rangle_6 \right)$  and  $E_* \left( \underline{BP} \langle 1 \rangle_4 \right)$  in order to define  $x_{(i)}$  and  $y_{(i)}$  as elements of  $E^* \left( \underline{BP} \langle 1 \rangle_4 \right)$  and  $E^* \left( \underline{BP} \langle 1 \rangle_6 \right)$ . There is a standard way to do this.

Given a monomial  $c \in E_* \left( \underline{BP} \langle 1 \rangle_r \right)$  in the elements  $[v_1]$ ,  $b_{(i)}$  and  $b_n$ , define  $Vc$  by replacing each  $b_{(i)}$  by  $b_{(i-1)}$  and each  $b_n$  by  $b_{n/2}$ , taking  $Vc = 0$  if any  $i = 0$  or any  $n$  is odd. (This depends on how  $c$  is written as a monomial, but the choice does not matter; the Verschiebung  $V$  is only defined naturally mod  $I_\infty$ .)

**Lemma 3.19** *For any  $y \in E^* \left( \underline{BP} \langle 1 \rangle_r \right)$  and monomial  $c$  as above,*

$$\langle y^2, c \rangle = \langle y, Vc \rangle^2.$$

**Proof.**  $\langle y^2, c \rangle = \langle \psi^*(y \otimes y), c \rangle = \langle y \otimes y, \psi c \rangle = \sum \langle y, c' \rangle \langle y, c'' \rangle$ , where  $\psi c = \Sigma c' \otimes c''$ . Suppose that

$$c = [v_1^m] \circ b_{i_1} \circ b_{i_2} \circ \cdots \circ b_{i_k}.$$

Then

$$\psi c = \sum [v_1^m] \circ b_{j_1} \circ b_{j_2} \circ \cdots \circ b_{j_k} \otimes [v_1^m] \circ b_{i_1-j_1} \circ b_{i_2-j_2} \circ \cdots \circ b_{i_k-j_k},$$

summing over all  $j_1, j_2, \dots, j_k$ ,  $0 \leq j_r \leq i_r$ , and

$$\langle y^2, c \rangle = \sum \langle y, [v_1^m] \circ b_{j_1} \circ b_{j_2} \circ \cdots \circ b_{j_k} \rangle \langle y, [v_1^m] \circ b_{i_1-j_1} \circ b_{i_2-j_2} \circ \cdots \circ b_{i_k-j_k} \rangle.$$

Since we are working mod 2, all terms cancel in pairs except for the desired middle term  $\langle y, Vc \rangle \langle y, Vc \rangle$ . The same proof works for  $*$ -products of such monomials  $c$ . ■

We construct a basis of  $E_* \left( \underline{BP} \langle 1 \rangle_6 \right)$  as follows. We have a basis of  $PE_* \left( \underline{BP} \langle 1 \rangle_6 \right)$  consisting of the elements  $b_{(0)} \circ b_{(i)}^{\circ 2}$  (for  $i \geq 0$ ) and  $b_{(0)} \circ \hat{b}_n$  (for  $n$  not a power of 2). We adjoin new basis elements  $b_{(k)} \circ b_{(k+i)}^{\circ 2}$  (for  $i \geq 0$  and  $k > 0$ ) and  $b_{(k)} \circ \hat{b}_{n,k}$  (for  $n$  not a power of 2 and  $k > 0$ ), where  $V^k \hat{b}_{n,k} = \hat{b}_n$ . The desired basis of  $E^* \left( \underline{BP} \langle 1 \rangle_6 \right)$  consists of all  $*$ -products of these basis elements (including the empty product 1), without repetition of factors.

We define  $y_{(i)}$  as dual to  $b_{(0)} \circ b_{(i)}^{\circ 2}$  (for  $i \geq 0$ ). The lemma tells us that, for all  $i, k \geq 0$ ,  $y_{(i)}^{2^k}$  is dual to  $b_{(k)} \circ b_{(k+i)}^{\circ 2}$ . (The duality for other monomials in the  $y$ 's is far more obscure.) We also define  $w_n$  as dual to  $b_{(0)} \circ \hat{b}_n$  (for  $n$  not a power of 2). Then we have  $E^* \left( \underline{BP} \langle 1 \rangle_6 \right) = B \hat{\otimes} N$ , where  $B = E^* [[y_{(i)} : i \geq 0]]$  and  $N = E^* [[w_n : n \neq 2^r]]$ .

A similar process extends the basis of  $PE_* \left( \underline{BP} \langle 1 \rangle_4 \right)$  to a basis of  $E_* \left( \underline{BP} \langle 1 \rangle_4 \right)$ . We define  $x_{(i)}$  as dual to  $b_{(0)} \circ b_{(i)}$  and  $u_n$  as dual to  $b_{(0)} \circ b_n$ . Then we have  $E^* \left( \underline{BP} \langle 1 \rangle_4 \right) = A \hat{\otimes} M$ , where  $A = E^* [[x_{(i)} : i \geq 0]]$  and  $M = E^* [[u_n : n \neq 2^r]]$ .

**Proposition 3.20** *We have a short exact sequence of (completed) Hopf algebras*

$$E^* \longleftarrow E^* (K(\mathbb{Z}, 3)) \xleftarrow{f^*} E^* [[y_{(i)} : i \geq 0]] \xleftarrow{v_1^*} E^* [[x_{(i)} : i \geq 0]] \longleftarrow E^*.$$

## 4 $E(1, 2)^* (K(\mathbb{Z}, 3))$

In this chapter, we assume that  $E$  is  $E(1, 2)$ .

We wish to compute  $v_1^* : A \hat{\otimes} M \longrightarrow B \hat{\otimes} N$ . This is more complicated than  $Qv_1^*$  because when we take decomposables into account,  $A$  no longer maps into  $B$ . We restate (6), this time including the decomposables.

**Lemma 4.1** *In  $E(1, 2)^* \left( \underline{BP} \langle 1 \rangle_6 \right)$ ,*

$$v_1^* (x_{(i)}) = v_1^{2^i} y_{(i)} + v_2^{2^i} y_{(i+1)} + \sum_{k \geq 2} A_{(k)}^{2^i} y_{(i+k)} + D_{(i)}$$

and

$$v_1^* (u_n) = w_n + \sum_j C_{n, 2^j} y_{(j)} + D_n$$

where  $D_{(i)}$  and  $D_n$  are decomposable. ■

It is easy to simplify the second equation. Because we are working in formal power series algebras, to make a valid change of generators, it is only necessary to check that it is invertible on the indecomposables. Here we simply replace  $w_n$  by  $w'_n = v_1^*(u_n)$ , so that the second equation becomes  $v_1^*(u_n) = w'_n$ . Then in the quotient algebra  $E(1, 2)^*(K(\mathbb{Z}, 3))$ , any occurrence of  $w'_n$  in  $D_{(i)}$  is replaced by zero.

First, we easily recover a well-known result.

**Proposition 4.2**  $K(2)^*(K(\mathbb{Z}, 3)) \cong K(2)^*[[y_{(0)}]]$ .

**Proof.** In this case, (6) simplifies to  $v_1^*(x_{(i)}) = v_2^{2^i} y_{(i+1)}$ . But,  $v_2$  is invertible. ■

## 4.1 The Bockstein Spectral Sequence

We have stable cofibrations

$$E(1, 2) \xrightarrow{v_1} E(1, 2) \longrightarrow K(2).$$

The induced long exact sequence is

$$\begin{aligned} \cdots &\longrightarrow E(1, 2)^*(K(\mathbb{Z}, 3)) \xrightarrow{v_1} E(1, 2)^*(K(\mathbb{Z}, 3)) \\ &\longrightarrow K(2)^*(K(\mathbb{Z}, 3)) \xrightarrow{\partial} E(1, 2)^*(K(\mathbb{Z}, 3)) \longrightarrow \cdots \end{aligned}$$

**Proposition 4.3** (In [RWY] Lemma 5.1 at Page 174) *Let  $X$  be a space. Let  $0 \leq k \leq n$  and  $n > 0$ . If  $K(n)^*(X)$  is even-dimensional, then  $E(k, n)^*(X)$  is even-dimensional and has no  $v_k$  torsion ( $v_0 = p$ ).*

According to the above section, we know that (ignoring the invertible element  $v_2$ )

$$\overline{K(2)^*}(K(\mathbb{Z}, 3)) \cong \mathbb{F}_2[[y_{(0)}]]$$

is even-dimensional. This implies  $K(2)^*(K(\mathbb{Z}, 3))$  is even-dimensional. Therefore,  $E(1, 2)^*(K(\mathbb{Z}, 3))$  is even-dimensional and has no  $v_1$  torsion. Hence, the long exact sequence breaks into the short exact sequence of graded groups,

$$0 \longrightarrow E(1, 2)^*(K(\mathbb{Z}, 3)) \xrightarrow{v_1} E(1, 2)^*(K(\mathbb{Z}, 3)) \longrightarrow K(2)^*(K(\mathbb{Z}, 3)) \xrightarrow{\partial} 0.$$

The Bockstein spectral sequence degenerates.

The filtration of the Bockstein spectral sequence provides a natural topology. We have the ideal in  $E(1, 2)^*(K(\mathbb{Z}, 3))$

$$(v_1^i) = \text{im} \left( E(1, 2)^*(K(\mathbb{Z}, 3)) \xrightarrow{v_1} E(1, 2)^*(K(\mathbb{Z}, 3)) \xrightarrow{v_1} \dots \xrightarrow{v_1} E(1, 2)^*(K(\mathbb{Z}, 3)) \right)$$

where we multiply by  $v_1^i$ . This gives us a decreasing filtration by ideals

$$E(1, 2)^*(K(\mathbb{Z}, 3)) = (v_1^0) \supseteq (v_1^1) \supseteq (v_1^2) \supseteq \dots,$$

where each quotient is a copy of  $K(2)^*(K(\mathbb{Z}, 3)) = K(2)^*[[y_{(0)}]]$ . In effect, this gives a lower bound for  $E(1, 2)^*(K(\mathbb{Z}, 3))$ . It gives no information on  $(v_1^\infty) = \bigcap_n (v_1^n)$ .

**Definition 4.4** *The topology on  $E(1, 2)^*(K(\mathbb{Z}, 3))$  from the Bockstein spectral sequence has a basis consisting of the cosets  $x + (v_1^i)$  for all  $x \in E(1, 2)^*(K(\mathbb{Z}, 3))$  and all  $i$ .*

## 4.2 Topology of Cohomology in $E(1, 2)^*(K(\mathbb{Z}, 3))$

We take  $E = E(1, 2)$  in this section.

**Definition 4.5** *Define  $z_{(i)} = f^*(y_{(i)})$  in  $E^*(K(\mathbb{Z}, 3))$  for all  $i \geq 0$ .*

We treat  $E^*(K(\mathbb{Z}, 3))$  as the cokernel (in the sense of algebras) of the algebra homomorphism  $v_1^* : A \rightarrow B$ , considered as a quotient of  $v_1^* : E^*(\underline{BP}\langle 1 \rangle_4) \rightarrow E^*(\underline{BP}\langle 1 \rangle_6)$ . Thus, the elements  $z_{(i)}$  generate  $E^*(K(\mathbb{Z}, 3))$  topologically. The formula for  $v_1^*(x_{(i)})$  provides the relation

$$v_1^{2^i} z_{(i)} + v_2^{2^i} z_{(i+1)} + \sum_{j \geq i+2} A_{(j-i)}^{2^i} z_{(j)} = D_{(i)},$$

where  $D_{(i)}$  is decomposable.

We have two topologies on  $E^*(K(\mathbb{Z}, 3))$ . The first, from the filtration by powers of the ideal  $(v_1)$ , was discussed in Section 4.1. This is the  $v_1$ -topology. We have good information on the quotients by  $(v_1^n)$ , but no information on  $(v_1^\infty)$ . The second is inherited from the topology on  $E^*(\underline{BP}\langle 1 \rangle_6)$  as dual to the algebra  $E_*(\underline{BP}\langle 1 \rangle_6)$ .

In the quotient by a basic neighborhood of 0, we retain only finitely many of the monomials in the  $z_{(i)}$ . We call this the  $z$ -topology.

We compare these two topologies on  $E^*(K(\mathbb{Z}, 3))$ . They are not the same. Let  $Z$  be the augmentation ideal generated topologically by all the  $z_{(i)}$ . Below, we display schematically the  $v_1$ -filtration and the filtration by (completed) powers of  $Z$ , where we treat  $\mathbb{F}_2[v_2, v_2^{-1}]$  as the graded ground field and suppress it.

$\cdots$	$z_{(0)}^3$	$z_{(0)}^2$	$z_{(0)}$	1
			$z_{(1)}$	$v_1$
		$z_{(1)}^2$		$v_1^2$
	$z_{(1)}^3$		$z_{(2)}$	$v_1^3$
			$\vdots$	$\vdots$

As with any path-connected space, there is a canonical  $E^*$ -module decomposition  $E^*(K(\mathbb{Z}, 3)) \cong E^*1 \oplus Z$ . It is obvious that no power of  $v_1$  lies in  $Z$ , and Section 4.1 shows that no power of  $z_{(0)}$  lies in  $(v_1)$ .

We need another coarser filtration that we call the  $z_+$ -filtration. We filter  $E^*(K(\mathbb{Z}, 3))$  by ideals generated topologically by all except finitely many monomials in the  $z_{(i)}$  for  $i > 0$ . This is again a complete filtration, being a quotient of the corresponding  $y_+$ -filtration on  $B$  indicated by writing  $B = (E^*[[y_{(0)}]])[[y_{(i)} : i > 0]]$ .

**Lemma 4.6** *The  $v_1$ -filtration and the  $z_+$ -filtration define the same topology on the augmentation ideal  $Z$ .*

**Proof.** A basic neighborhood  $U$  of 0 in the  $z_+$ -filtration is the ideal generated (topologically) by the elements  $z_{(i)}$  for  $i \geq k$  and  $Z^m$ , for some  $k$  and  $m$ . We saw already in the proof of Theorem 3.18 that  $v_1^{2^k-1}z_{(i)} \in U + Z^2$  for all  $i$ , i.e.,  $v_1^{2^k-1}Z \subset U + Z^2$ . Then  $v_1^{2^k-1}Z^j \subset U + Z^{j+1}$  for all  $j$ , and

$$\begin{aligned} v_1^{(m-1)(2^k-1)}Z &\subset U + v_1^{(m-2)(2^k-1)}Z^2 \subset U + v_1^{(m-3)(2^k-1)}Z^3 \subset \cdots \\ &\subset U + v_1^{2^k-1}Z^{m-1} \subset U + Z^m = U. \end{aligned}$$

Conversely, given  $n$ , we have seen that  $z_{(i)} \in (v_1^n)$  if  $2^{i-1} \geq n$  and that  $z_{(i)}^n \in (v_1^n)$  for all  $i > 0$ . Thus the  $z_+$ -neighborhood  $U$  lies in  $(v_1^n)$  if we take  $m = n$  and choose  $k$  such that  $2^{k-1} \geq n$ . ■

**Remark 4.7** The  $v_1$ -filtration is clearly not complete in the summand  $E^*1$ . Both the  $z$ - and  $z_+$ -filtrations are complete, but make the  $E^*1$  summand discrete.

**Corollary 4.8** *The  $z$ -filtration and the filtration consisting of the ideals  $((v_1^i) \cap Z) + (z_{(0)}^j)$  for  $i, j \geq 0$  define the same topology on  $E^*(K(\mathbb{Z}, 3))$ .*

**Proof.** A basic  $z$ -neighborhood of 0 has the form  $U + (z_{(0)}^j)$ , where  $U$  is a  $z_+$ -neighborhood of 0. Choose  $i$  such that  $(v_1^i) \cap Z \subset U$ . Then  $((v_1^i) \cap Z) + (z_{(0)}^j) \subset U + (z_{(0)}^j)$ .

Conversely, given an ideal  $((v_1^i) \cap Z) + (z_{(0)}^j)$ , we can find a  $z_+$ -neighborhood  $U$  of 0 such that  $U \subset (v_1^i) \cap Z$ . Then we have a  $z$ -neighborhood  $U + (z_{(0)}^j) \subset ((v_1^i) \cap Z) + (z_{(0)}^j)$ . ■

**Theorem 4.9** *As a topological ring,  $E(1, 2)^*(K(\mathbb{Z}, 3)) \cong E(1, 2)^*[[z_{(0)}]]^\wedge$ , where the completion indicates  $v_1$ -completion except on the  $E^*$ -module summand  $E^*1$  generated by 1. Precisely,*

$$E(1, 2)^*(K(\mathbb{Z}, 3)) \cong E^*1 \oplus \mathbb{F}_2[v_2, v_2^{-1}][[v_1, z_{(0)}]]z_{(0)}.$$

**Proof.** By Lemma 4.6,  $Z$  is complete under the  $v_1$ -filtration. Section 4.1 shows that  $E^*(K(\mathbb{Z}, 3)) / (v_1^n) \cong \mathbb{F}_2[v_2, v_2^{-1}][[z_{(0)}]][v_1] / (v_1^n)$ . Hence

$$\begin{aligned} Z &= \lim_n Z / v_1^n Z = \left( \lim_n \mathbb{F}_2[v_2, v_2^{-1}][[z_{(0)}]][v_1] / (v_1^n) \right) z_{(0)} \\ &= \mathbb{F}_2[v_2, v_2^{-1}][[z_{(0)}]][[v_1]]z_{(0)}. \end{aligned}$$

The topology on  $E^*(K(\mathbb{Z}, 3))$  is given by the  $z$ -filtration, or, in view of Corollary 4.8, by the ideals  $((v_1^i) \cap Z) + (z_{(0)}^j)$ . We can now rewrite  $(v_1^i) \cap Z$  as  $(v_1^i z_{(0)})$ . The summand  $E^*1$  is simply a copy of  $E^*$ , with no completion. ■

### 4.3 Relations in $E(1, 2)^*(K(\mathbb{Z}, 3))$

Furthermore, we try to generalize Tamanoi's result to higher degree. Therefore, we change our generators a little to get a better looking formula. We note that we do not use the invertibility of  $v_2$ . Again, we write  $E = E(1, 2)$ .

Earlier, we showed that in  $E^* \left( \underline{BP} \langle 1 \rangle_6 \right)$ ,

$$v_1^* (x_{(i)}) = v_1^{2^i} y_{(i)} + v_2^{2^i} y_{(i+1)} + \sum_{j \geq i+2} A_{(j-i)}^{2^i} y_{(j)} + D_{(i)}$$

for  $i \geq 0$ , with  $D_{(i)}$  decomposable. It would be desirable to change generators to eliminate the unwanted terms. We can do the following:

**Theorem 4.10** *There are alternate generators  $y'_{(i)}$  of the algebra  $B$  such that  $B = E^* \left[ \left[ y'_{(i)} : i \geq 0 \right] \right]$  and*

$$v_1^* (x_{(i)}) = v_1^{2^i} y'_{(i)} + v_2^{2^i} y'_{(i+1)} + D_{(i)}$$

for all  $i \geq 0$ , where  $D_{(i)}$  is decomposable.

Instead of changing generators directly, it is more convenient to change the basis of  $QE_* \left( \underline{BP} \langle 1 \rangle_6 \right)$  before dualizing; then the change of generators will automatically be continuous. First, we need to improve a previous observation.

**Lemma 4.11** *For  $k \geq 2$ ,  $A_{(k)}$  is divisible by  $v_1^2$ .*

**Proof.** Recall from Lemma 2.10 that  $A_{(k)}$  is the coefficient of  $x^{2^{k+1}}$  in

$$[2](x) = v_1 x^2 +_F v_2 x^4 = v_1 x^2 + v_2 x^4 + \sum_{i,j \geq 1} a_{ij} v_1^i v_2^j x^{2i+4j}.$$

Modulo  $v_1^2$ , the only terms that survive are  $v_1 x^2 + v_2 x^4 + \sum_{j \geq 1} a_{1j} v_1 v_2^j x^{4j+2}$ . There is no term in  $x^{2^n}$  for  $n \geq 3$ . ■

We consider  $Qv_{1*} : QE_* \left( \underline{BP} \langle 1 \rangle_4 \right) \longrightarrow QE_* \left( \underline{BP} \langle 1 \rangle_2 \right)$ , as before. Again, we suppress  $\circ$  here, as it is the only multiplication.

**Lemma 4.12** *In  $QE_* \left( \underline{BP} \langle 1 \rangle_4 \right)$ , there are elements  $d_{(i)}$  and  $e_{(i)}$  such that:*

(a)  $v_{1*} (d_{(i)}) = v_1^{2^{i+1}-1} b_{(i)}$  for all  $i \geq 0$ .

(b) The elements  $e_{(i)}$  for all  $i \geq 0$  form a basis, and satisfy

$$v_{1*} (e_{(i)}) = v_1^{2^i} b_{(i)} + v_2^{2^{i-1}} b_{(i-1)}.$$

**Proof.** We construct  $d_{(i)}$  and  $e_{(i)}$  inductively, starting from  $d_{(0)} = e_{(0)} = b_{(0)}^2$ .

Suppose we have  $d_{(i)}$  for  $i < k$ . We recall from (6) that

$$v_{1*} (b_{(k)}^2) = v_1^{2^k} b_{(k)} + v_2^{2^{k-1}} y_{(k-1)} + \sum_{i \geq 2} A_{(i)}^{2^{k-i}} y_{(k-i)}.$$

By induction,

$$v_{1*} \left( \left( v_1^{-2^{k-i+1}+1} A_{(i)}^{2^{k-i}} \right) d_{(k-i)} \right) = A_{(i)}^{2^{k-i}} b_{(k-i)},$$

where we note that the coefficient of  $d_{(k-i)}$  lies in  $E_*$ . We set

$$e_{(k)} = b_{(k)}^2 + \sum_{i \geq 2} \left( v_1^{-2^{k-i+1}+1} A_{(i)}^{2^{k-i}} \right) d_{(k-i)},$$

to cancel out the unwanted terms. To remove the  $b_{(k-1)}$  term also, we multiply by a further  $v_1^{2^k-1}$  and set  $d_{(k)} = v_1^{2^k-1} e_{(k)} + v_2^{2^{k-1}} d_{(k-1)}$ , so that

$$v_{1*} (d_{(k)}) = v_1^{2^k-1+2^k} b_{(k)} + v_1^{2^k-1} v_2^{2^{k-1}} b_{(k-1)} + v_1^{2^k-1} v_2^{2^{k-1}} b_{(k-1)} = v_1^{2^{k+1}-1} b_{(k)}.$$

■

**Remark 4.13** Closer examination of the proof shows that the elements  $e_{(i)}$  are uniquely determined by property (b).

**Proof of Theorem.** We proceed as before. We use the elements  $b_{(0)} \circ e_{(i)}$  as a basis of  $PE_* \left( \underline{BP} \langle 1 \rangle_6 \right)$ , instead of the  $b_{(0)} \circ b_{(i)}^{\circ 2}$ , and extend to a basis of the whole of  $E_* \left( \underline{BP} \langle 1 \rangle_6 \right)$ . We define  $y'_{(i)}$  as dual to  $b_{(0)} \circ e_{(i)}$ . The simplified formula for  $v_{1*} (e_{(i)})$  leads to the simplified formula for  $v_1^* (y'_{(i)})$ . ■

**Remark 4.14** This leads to a more transparent interpretation of Theorem 4.9. Put  $z'_{(i)} = f^* (y'_{(i)})$  for each  $i$ . In terms of these generators of  $E^* (K(\mathbb{Z}, 3))$ , the relations become

$$v_1^{2^i} z'_{(i)} + v_2^{2^i} z'_{(i+1)} = D_{(i)}.$$

Modulo decomposables,  $z'_{(i+1)} \equiv v_1^{2^i} v_2^{-2^i} z'_{(i)}$ , so by induction

$$z'_{(n)} \equiv v_1^{2^n-1} v_2^{-2^n+1} z'_{(0)}$$

for all  $n$ .

**Remark 4.15** It would be desirable to find a nonlinear change of generators that would remove the terms  $D_{(i)}$  as well, but this appears far more difficult.

## References

- [A71] J.F. Adams, *Stable Homotopy and Generalized Homology*, University of Chicago Press, 1971.
- [Brown] E.H. Brown, Jr., *Cohomology Theories*, Ann. of Math., 75(1962), 467-484; with a correction, Ann. of Math., 78(1963), 201.
- [BP66] E.H. Brown, Jr. and F.P. Peterson, *A Spectrum whose  $\mathbb{Z}_p$  Cohomology Is the Algebra of Reduced  $p^{\text{th}}$  Powers*, Topology, 5(1966), 149-154.
- [B82] J. Michael Boardman, *The Eightfold Way to BP-operations or  $E_*E$  and All That*, Canadian Math. Soc. Conf. Proc., 2(1982), Part 1, 187-226.
- [B95] J. Michael Boardman, *Stable Operations in Generalized Cohomology*, In I.M. James, editor, *The Handbook of Algebraic Topology*, Chapter 14, Elsevier, Amsterdam, 1995, 585-686.
- [BJW] J. Michael Boardman, David C. Johnson and W. Stephen Wilson, *Unstable Operations in Generalized Cohomology*, In I.M. James, editor, *The Handbook of Algebraic Topology*, Chapter 15, Elsevier, Amsterdam, 1995, 687-828.
- [G] P.G. Goerss, *Hopf Rings, Dieudonné Module, and  $E_*\Omega^2S^3$* , In J.-P. Meyer, J. Morava and W.S. Wilson, editors, *Homotopy Invariant Algebraic Structures: a Conference in Honor of J. Michael Boardman*, Contemporary Mathematics, Providence, R.I., 1999, AMS.
- [HRW] M.J. Hopkins, D.C. Ravenel and W.S. Wilson, *Morava Hopf Algebra and Spaces  $K(n)$  Equivalent to Finite Postnikov Systems*, In S.O. Kochman and P. Selick, editors, *Stable and Unstable Homotopy*, Fields Institute Communications, Providence, R.I., 1997, AMS.
- [HT] J.R. Hunton and P.R. Turner, *Coalgebraic Algebra*, J. of Pure and Applied Algebra, 129(1998), 297-313.

- [JW] D.C. Johnson and W.S. Wilson, *Projective Dimension and Brown-Peterson Homology*, *Topology*, 12(1973), 327-353.
- [Ka] T. Kashiwabara, *On Brown-Peterson Cohomology of  $QX$* , *Annals of Mathematics*, 153(2)(2001), 297-328.
- [KaW] T. Kashiwabara and W.S. Wilson, *The Morava  $K$ -theory and Brown-Peterson Cohomology of Spaces Related to  $BP$* , *Journal of Mathematics of Kyoto University*, 41(1)(2001), 43-95.
- [M58] J.W. Milnor, *The Steenrod Algebra and Its Dual*, *Ann. of Math.*, 67(1958), 150-171.
- [M62] J.W. Milnor, *On Axiomatic Homology Theory*, *Pacific J. Math.*, 12(1962), 337-341.
- [MM] J.W. Milnor and J.C. Moore, *On the structure of Hopf algebras*, *Ann. of Math.*, 81(1965), 211-264.
- [MT] R.E. Mosher and M.C. Tangora, *Cohomology Operations and Applications in Homotopy Theory*, Harper & Row, Publishers, New York, 1968.
- [Q] D. Quillen, *On the formal group laws of unoriented and complex cobordism theory*, *Bull. Amer. Math. Soc.*, 75(1969), 1293-1298.
- [RW77] D.C. Ravenel and W.S. Wilson, *The Hopf Ring for Complex Cobordism*, *J. of Pure and Applied Algebra*, 9(1977), 241-280.
- [RW80] D.C. Ravenel and W.S. Wilson, *Morava  $K$ -theory of Eilenberg-MacLane Spaces and the Conner-Floyd Conjecture*, *Amer. J. Math.*, 102(1980), 691-748.
- [RWY] D.C. Ravenel, W.S. Wilson and N. Yagita, *Brown-Peterson Cohomology from Morava  $K$ -theory*, *K-Theory*, 12(2)(1998), 149-200.
- [SW] H. Sadofsky and W.S. Wilson, *Commutative Morava Homology Hopf Algebras*, *Contemporary Mathematics*, 220(1998), 367-373.

- [S] L. Smith, *Lectures on the Eilenberg-Moore spectral sequence*, Lecture Notes in Mathematics, 134, Springer-Verlag.
- [T] H. Tamanoi, *Spectra of BP-linear Relations,  $v_n$ -series, and BP Cohomology of Eilenberg-Mac Lane Spaces*, Trans. Amer. Math. Soc., 352(2000), no.11, 5139-5178. .
- [Wil73] W.S. Wilson, *The  $\Omega$ -spectrum for Brown-Peterson Cohomology, Part I*, Comment Math. Helv., 48(1973), 45-55.
- [Wil75] W.S. Wilson, *The  $\Omega$ -spectrum for Brown-Peterson Cohomology, Part II*, Amer. J. Math., 97(1975), 101-123.

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