PRACTICE EXAM

1. Midterm 1 Material

1.1. **Differential Equation.**

1.1.1. *#1.* Solve the following differential equation when $x = \sqrt{\frac{3}{5}}$ and $y = 1$.

$$\frac{dy}{dx} = -4y^4x$$

Solution:
This is a separable equation and we have

$$\int y^{-4}dy = -4\int xdx$$
$$\frac{y^{-5}}{-5} = -2x^2 + C$$

Plugging in the initial conditions we have

$$-\frac{1}{5} = -2\cdot\frac{3}{5} + C$$
$$1 = C$$

and so

$$y^{-5} = 10x^2 - 5$$

and finally

$$y = \frac{1}{\sqrt[5]{10x^2 - 5}}$$

It's important to solve for y, unless absolutely impossible.

1.1.2. *#2.* Find the equilibrium points for the differential equation.

$$\frac{dy}{dx} = (y+1)(y-1)(y-2)$$

1.1.3. *#3.* Which equilibrium points are stable and which are unstable?
Solution to #2:
$y = -1$, $y = 0$, and $y = 1$.
Solution to #3:
This is a third degree polynomial with positive leading coefficient. There are no repeating roots. We immediately know (or we can also check by plugging in points) that the function is

$$\begin{cases} \text{less than } 0 & \text{if } y < -1 \\ \text{greater than } 0 & \text{if } -1 < y < 1 \\ \text{less than } 0 & \text{if } 1 < y < 2 \\ \text{greater than } 0 & \text{if } 2 < y \end{cases}$$

Thus $y = -1$ and $y = 2$ are unstable equilibrium point and $y = 1$ is a stable equilibrium point.

1.2. **Leslie Matrices.**

1.2.1. *#4.* We have a population of newborns, N_0, and one-year olds, N_1. There are no two-year olds or older. One third of the newborns survive to the next year to be one-year olds. Each newborn produces 1 newborn for the next year.. Each one-year old produces 6 newborns for the next year. What is the Leslie matrix that takes

$$\begin{bmatrix} N_0(t) \\ N_1(t) \end{bmatrix} \quad \text{to} \quad \begin{bmatrix} N_0(t+1) \\ N_1(t+1) \end{bmatrix}?$$

1.2.2. *#5.* Find the eigenvalues.

1.2.3. *#6.* Find the associated eigenvectors.

1.2.4. *#7.* What is a stable age distribution $\binom{a}{b}$ for this population?

Solution to #4:

We have

$$\begin{aligned} \begin{bmatrix} N_0(t+1) \\ N_1(t+1) \end{bmatrix} &= \begin{bmatrix} 1N_0(t) + 6N_1(t) \\ \frac{1}{3}N_0(t) \end{bmatrix} \\ &= \begin{bmatrix} 1 & 6 \\ \frac{1}{3} & 0 \end{bmatrix} \begin{bmatrix} N_0(t) \\ N_1(t) \end{bmatrix}. \end{aligned}$$

Thus the Leslie Matrix is

$$\begin{bmatrix} 1 & 6 \\ \frac{1}{3} & 0 \end{bmatrix}.$$

Solution to #5:

We have

$$\begin{aligned} \det(A - \lambda I) &= \begin{vmatrix} 1-\lambda & 6 \\ \frac{1}{3} & 0-\lambda \end{vmatrix} \\ &= \lambda^2 - \lambda - 2 \\ &= (\lambda - 2)(\lambda + 1) \end{aligned}$$

So $\lambda = 2$,$\lambda = -1$ are eigenvalues.

Solution to #6:

$\lambda = 2:$

$$(A - 2I)v = \begin{bmatrix} -1 & 6 \\ \frac{1}{3} & -2 \end{bmatrix} v = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

implies

$$v = \begin{bmatrix} 6 \\ 1 \end{bmatrix}.$$

$\lambda = -1$:

$$(A - (-1)I)v = \begin{bmatrix} 2 & 6 \\ \frac{1}{3} & 1 \end{bmatrix} v = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

implies

$$v = \begin{bmatrix} 3 \\ -1 \end{bmatrix}.$$

Solution to #7:

A stable age distribution is an eigenvector corresponding to the larger eigenvalue. Thus a valid stable age distribution is

$$\begin{bmatrix} 6 \\ 1 \end{bmatrix}$$

2. Midterm 2 Material

2.1. **System of Differential Equations.** #8,#9 Solve the system of differential equations:

$$\begin{aligned} x'(t) &= 2x(t) + y(t) \\ y'(t) &= 7x(t) - 4y(t) \end{aligned}$$

Find the eigenvalues, Find the eigenvectors.

Solution:

The system can be written as

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 7 & -4 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

and we start by solving the eigenvalues and eigenvectors of the corresponding matrix

$$\begin{bmatrix} 2 & 1 \\ 7 & -4 \end{bmatrix}.$$

We have

$$\begin{aligned} \det(A - \lambda I) &= \begin{vmatrix} 2-\lambda & 1 \\ 7 & -4-\lambda \end{vmatrix} \\ &= \lambda^2 + 2\lambda - 15 \\ &= (\lambda - 3)(\lambda + 5) \end{aligned}$$

So $\lambda = 3$,$\lambda = -5$ are eigenvalues.

$\lambda = 3$:

$$(A - 3I)v = \begin{bmatrix} -1 & 1 \\ 7 & -7 \end{bmatrix} v = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

implies

$$v = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

$\lambda = -5$:

$$(A - (-5)I)v = \begin{bmatrix} 7 & 1 \\ 7 & 1 \end{bmatrix} v = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

implies

$$v = \begin{bmatrix} 1 \\ -7 \end{bmatrix}.$$

The general answer is then

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1 e^{\lambda_1 t} v_1 + c_2 e^{\lambda_2 t} v_2$$

for some constants c_1 and c_2. Then the answer to this problem is

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1 e^{3t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-5t} \begin{bmatrix} 1 \\ -7 \end{bmatrix}.$$

#10 Suppose the previous question had the initial conditions

$$\begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} 5 \\ -3 \end{bmatrix}.$$

Solve the initial value problem. (That is, solve for the c_1 and c_2)

Solution:

Plugging in $t = 0$, we have

$$\begin{bmatrix} 5 \\ -3 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -7 \end{bmatrix}.$$

Solving this system of equations, we get $c_1 = 4$ and $c_2 = 1$. Thus the solution to the initial value problem is

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = 4e^{3t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + e^{-5t} \begin{bmatrix} 1 \\ -7 \end{bmatrix}$$

#11 What type of equilibrium point is at $(0, 0)$?

Solution:

The associated matrix has a positive and negative eigenvalue, so the equilibrium point at $(0, 0)$ is a saddle point.

2.2. **Questions that are Asked of a Function with Two Variables.** That's right... TWO!

$$f(x, y) = 4x^2 + 2x + y^2$$

#12 Compute the gradient of f, i.e. ∇f.

Solution:

$$\nabla f(x, y) = \begin{bmatrix} \frac{\partial f}{\partial x}(x, y) \\ \frac{\partial f}{\partial y}(x, y) \end{bmatrix} = \begin{bmatrix} 8x + 2 \\ 2y \end{bmatrix}$$

When simply asked for the gradient, do not evaluate at any point.

#13 Compute the gradient at the point $(1, 2)$. Same question: Compute the direction of maximum slope at $(1, 2)$.

Solution:

We use our answer from #12 and plug in the point $(x, y) = (1, 2)$. We get

$$\nabla f(1, 2) = \begin{bmatrix} 10 \\ 4 \end{bmatrix}.$$

#14 Compute the slope of f in the direction $(-3, 4)$ at the point $(1, 2)$.

Solution:

We can find this by taking the directional derivative of f in the direction $(-3, 4)$. We normalize $(-3, 4)$ and get

$$u = \frac{1}{\sqrt{5}} \begin{bmatrix} -3 \\ 4 \end{bmatrix}$$

so that

$$\begin{aligned} \nabla f(1, 2) \cdot u &= \frac{1}{\sqrt{5}} \begin{bmatrix} 10 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} -3 \\ 4 \end{bmatrix} \\ &= \frac{1}{\sqrt{5}} (-30 + 16) = \frac{-14}{\sqrt{5}} \end{aligned}$$

#15 Compute the slope of f at $(1,2)$ in the direction it is maximal.
Solution:
This problem is like the last, except in the direction $(10,4)$. We normalize $(10,4)$ and get

$$u = \frac{1}{\sqrt{116}} \begin{bmatrix} 10 \\ 4 \end{bmatrix}$$

so that

$$\begin{aligned} \nabla f(1,2) \cdot u &= \frac{1}{\sqrt{116}} \begin{bmatrix} 10 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} 10 \\ 4 \end{bmatrix} \\ &= \frac{1}{\sqrt{116}}(100+16) = \sqrt{116}. \end{aligned}$$

$$f(x,y) = 4x^2 + 2x + y^2$$

#16 Find the equation for the tangent plane to f at the point $(1,2)$. Place the solution in the form

$$z - z_0 = a(x - x_0) + b(y - y_0).$$

Solution:
The equation of the tangent plane at the point (x_0, y_0) is given by

$$z - f(x_0,y_0) = \frac{\partial f}{\partial x}(x_0,y_0) \cdot (x - x_0) + \frac{\partial f}{\partial y}(x_0,y_0) \cdot (y - y_0).$$

We have

$$\begin{aligned} f(1,2) &= 10, \\ \frac{\partial f}{\partial x}(1,2) &= 10, \end{aligned}$$

and

$$\frac{\partial f}{\partial y}(1,2) = 4$$

so that the equation of the tangent plane at $(1,2)$ is given by

$$z - 10 = 10(x-1) + 4(y-2)$$

#17 Give a parametric equation for the tangent line to the graph of f (in 3-space) for $(x,y) = (1,2)$ in the direction $(-3,4)$.
Solution:
Using the information already provided, the answer is

$$\begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 10 \end{bmatrix} + t \begin{bmatrix} -3/\sqrt{5} \\ 4/\sqrt{5} \\ -14/\sqrt{5} \end{bmatrix}.$$

To make the answer look nice, we can multiple the direction of the line by a scalar multiple. Thus, we have an equivalent representation of the line as

$$\begin{bmatrix} x(r) \\ y(r) \\ z(r) \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 10 \end{bmatrix} + r \begin{bmatrix} -3 \\ 4 \\ -14 \end{bmatrix}.$$

Let's consider the more general question. Give a parametric equation for the tangent line to the graph of f for $(x,y) = (x_0,y_0)$ in the direction (v_1,v_2). First we normalize $v = (v_1, v_2)$ and get $u = (u_1, u_2)$. Then by ensuring the line belongs to the tangent plane, we obtain the following solution:

$$\begin{bmatrix} x(s) \\ y(s) \\ z(s) \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \\ f(x_0, y_0) \end{bmatrix} + s \begin{bmatrix} u_1 \\ u_2 \\ \nabla f(x_0, y_0) \cdot u \end{bmatrix}.$$

#18 Compute the Hessian of f.

Solution:

We compute the second derivatives of f and obtain

$$\frac{\partial^2 f}{\partial x^2}(x, y) = 8,$$

$$\frac{\partial^2 f}{\partial x \partial y}(x, y) = \frac{\partial^2 f}{\partial y \partial x}(x, y) = 0,$$

and

$$\frac{\partial^2 f}{\partial y^2}(x, y) = 2,$$

so that

$$\text{Hess}[f](x, y) = \begin{bmatrix} 8 & 0 \\ 0 & 2 \end{bmatrix}.$$

#19 Find the one critical point for f.

The critical point for f is found by setting the gradient equal to zero. From

$$\begin{bmatrix} 8x + 2 \\ 2y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

we solve and get $(x, y) = \left(-\frac{1}{4}, 0\right)$.

$$f(x, y) = 4x^2 + 2x + y^2$$

#20, #21, #22, #23 Consider the function on the domain $x^2 + y^2 \leq 1$. What is the minimum value f takes and where? What is the maximum f takes and where?

Solution to #20, #21 (Geometric):

Geometrically we know f is an upward facing paraboloid. Thus it takes its minimum at its critical point $\left(-\frac{1}{4}, 0\right)$. We have

$$f\left(-\frac{1}{4}, 0\right) = 4 \cdot \frac{1}{16} - \frac{1}{2} + 0 = -\frac{1}{4}$$

and so the minimum value is $-\frac{1}{4}$.

Solution to #22, #23 (Geometric):

Continuing our geometric viewpoint, the point (x, y) furthest from the center will give the maximum. This occurs at $(1, 0)$. We have

$$f(1, 0) = 4 + 2 + 0 = 6$$

and so the maximum value is 6.

Solution (Non-Geometric):

A combination of #18 and #19 gives that f has a local minimum at $\left(-\frac{1}{4}, 0\right)$. We have

$$f\left(-\frac{1}{4}, 0\right) = 4 \cdot \frac{1}{16} - \frac{1}{2} + 0 = -\frac{1}{4}.$$

Along the boundary we use Lagrange multipliers. Let $g(x, y) = x^2 + y^2 - 1$. Then setting

$$\nabla f = \lambda \nabla g$$

we have

$$\begin{bmatrix} 8x+2 \\ 2y \end{bmatrix} = \lambda \begin{bmatrix} 2x \\ 2y \end{bmatrix}$$

and so

$$\begin{aligned} 8x + 2 &= \lambda 2x \\ 2y &= \lambda 2y. \end{aligned}$$

The second equation implies $y = 0$ or $\lambda = 1$. If $y = 0$, then using the constraint we get $x = \pm 1$. If $\lambda = 1$ then $x = -1$ and using the constraint we get $y = 0$.

Evaluating f as these points, we have

$$f(-1, 0) = 4 - 2 + 0 = 2$$

and

$$f(1, 0) = 4 + 2 + 0 = 6$$

so we conclude the minimum occurs at $\left(-\frac{1}{4}, 0\right)$ with value $-\frac{1}{4}$ and the maximum occurs at $(1, 0)$ with value 6.

3. Newest Material - Probability and Statistics

3.1. **Sandwich.** #24 Suppose you have three types of bread (white, wheat, and pita), two types of protein (mushroom and turkey), two types of extras (pickle and tomato), and three types of cheese (provolone, American, and cheddar). Using one type of bread on either end and concerned about the order of ingredients, how many sandwiches can you make?

Solution:

We have bread+(protein/extras/cheese)+bread. If we can determine the ways to combine (protein/extras/cheese) then we can just multiply by the three types of bread. We might think the different orders of the ingredients numbers 6:

$$\{PEC, PCE, CPE, CEP, ECP, EPC\}.$$

However, we can flip the sandwich over to get a combination we already have. Thus the unique orders are just 3 (half of the 6):

$$\{PEC, PCE, CPE\}.$$

3.2. **Counting.** #25 Suppose there are nine females and six males at a dance. How many ways are there to form four couples on the dance floor?

Solution:

At first, let's compute the number of ways to form four ordered couples. We'd have

$$9 \cdot 8 \cdot 7 \cdot 6$$

choices for females and

$$6 \cdot 5 \cdot 4 \cdot 3$$

choices for males. However, once this gives us four couples, we have to remove the ordering of the four couples. Thus, we divide by 4!.

After multiplying and dividing, we have

$$9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 3$$

combinations.

3.3. **Standard Deck of Cards.** #26a What's the probability of getting a full house when picking five cards? A full house is a three of a kind and a pair.

Solution:

First let's fix a number for the three of a kind and a pair. We have 13 values for the first and then 12 values for the second (or vice versa). In the end, we multiply our result by $13 \cdot 12$. Once we've fixed a number, we count how many ways we could have chosen that number to make three of a kind. That is, four choose three: $\binom{4}{3}$. And for the pair we have four choose two: $\binom{4}{2}$. In total, we have $\binom{13}{1}\binom{4}{3}\binom{12}{1}\binom{4}{2}$ possibilities for full houses. While the total number of outcomes ways to choose five cards from fifty-two cards, that is fifty-two choose five $\binom{52}{5}$. We conclude the probability of a full house is

$$\frac{13 \cdot 12\binom{4}{3}\binom{4}{2}}{\binom{52}{5}}.$$

For the exam, we leave the answer like that, but this turns out to be approximately 0.00144...

Try your "hand" at other Poker hands.

#26b Royal Flush (10 J Q K A of the same suit).

Solution to #26b:

We choose one of four suits and then the hand we must get is unique. so

$$\binom{4}{1}.$$

Divide by $\binom{52}{5}$.

#26c Straight Flush (five consecutive cards of the same suit). This includes the straight starting with an ace-low, that is A2345. For simplicity, we'll include the Royal Flush.

Solution to #26c:

We choose one of four suits and then choose a low card. The choices are $\{A, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. Thus we have

$$\binom{4}{1}\binom{10}{1}.$$

Divide by $\binom{52}{5}$.

#26d Four of a kind

Solution to #26d:

We choose one of 13 values for the four of a kind and then choose one of the remaining cards for the single.

$$\binom{13}{1}\binom{48}{1}.$$

Divide by $\binom{52}{5}$.

#26e Flush (five cards of the same suit). For simplicity, we'll include Royal Flushes and Straight Flushes.

Solution to #26e:

We choose one of four suits and then choose five of the thirteen possible from that suit. Thus we have

$$\binom{4}{1}\binom{13}{5}.$$

Divide by $\binom{52}{5}$.

#26f Straight (five consecutive cards). This includes the straight starting with an ace-low, that is A2345. For simplicity, we'll include the Royal Flush.

Solution to #26f:

We choose the starting low card. There are one in ten choices (see #25c). Then, because there is no restriction to suit, for each card we can pick one among four choices. Thus we have

$$\binom{10}{1}\binom{4}{1}^5.$$

Divide by $\binom{52}{5}$.

#26g Skipping three of a kind, we consider two pair and exclude the case of three of a kind and four of a kind. Thus the two pairs are distinct and there's a fifth card different from the values of the two pairs.

Solution to #26g:

We choose two values from the thirteen available for the pairs. Then from one of the values we choose two, from a second value we choose two. We choose one value from the remaining eleven for the single and from that choice we choose one of four. We have

$$\binom{13}{2}\binom{4}{2}\binom{4}{2}\binom{11}{1}\binom{4}{1}.$$

Divide by $\binom{52}{5}$.

3.4. **Sample Space.** #27 Determine the sample space for flipping a coin three times. What is the probability of getting at least two heads?

Solution:

$$S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$$

There are different ways to organize the solution to ensure you list all the possibilities. At the least, you should compute the possibilities and make sure you have the right number. In this case, I already know I should have eight possibilities, so after I'm done, I at least make sure I have eight elements in my sample space.

Of the events listed, the following have at least two heads:

$$\{HHH, HHT, HTH, THH\}.$$

Thus the probability of getting at least two heads is $\frac{4}{8}$ or simply $\frac{1}{2}$.

3.5. **Calculating Expected Value.** #28 Suppose you meet someone on the street telling you that he'll roll a die and if it comes up odd you have to pay a two dollars, but it if it comes up a two or four you will win a quarter and if it comes up a six you will win five dollars. If you play, what is your expected value?

Solution:

Let X be the value of your earnings in dollars. Then X takes the values -2, 0.25, and 5. Correspondingly, $\mathcal{P}(X = -2) = \frac{1}{2}$, $\mathcal{P}(X = 0.25) = \frac{1}{3}$, and $\mathcal{P}(X = 5) = \frac{1}{6}$.

Then we have

$$\begin{aligned}\mathbb{E}(X) &= \sum_x x\mathcal{P}(X=x)\\ &= -2\mathcal{P}(X=-2) + (0.25)\mathcal{P}(X=0.25) + 5\mathcal{P}(X=5)\\ &= -2\cdot\frac{1}{2} + 0.25\cdot\frac{1}{3} + 5\cdot\frac{1}{6}\\ &= -1 + \frac{1}{12} + \frac{5}{6} = -\frac{1}{12}.\end{aligned}$$

The expected value is $-\frac{1}{12}$. In other words, every time you play, you would expect to lose 8.3 cents.

3.6. **Colored Balls.** #29 Suppose you have four green balls, five red balls, and three blues balls. Pick seven balls with replacement (take, note the color, replace, repeat five times). What is the probability of having the number of red balls be greater than the number of blue balls by one?

Solution:

At first, we ask, when does this happen? Writing out the possibilities, we come up with the following:

Blue	Red	Green
1	2	4
2	3	3
3	4	0

Using the multinomial distribution, we have come up with the following probability:

$$\frac{7!}{1!2!4!}\left(\frac{4}{12}\right)^1\left(\frac{5}{12}\right)^2\left(\frac{3}{12}\right)^4+\frac{7!}{2!3!3!}\left(\frac{4}{12}\right)^2\left(\frac{5}{12}\right)^3\left(\frac{3}{12}\right)^3+\frac{7!}{3!4!0!}\left(\frac{4}{12}\right)^3\left(\frac{5}{12}\right)^4\left(\frac{3}{12}\right)^0$$

#30 Suppose you have eight green balls, four red balls, and nine blues balls. Pick three balls without replacement. What is the probability you get three balls from two different color groups?

Solution:

This situation occurs when you choose two from one color and a third ball of a different color. Thus for green and another color we have

$$\binom{8}{2}\binom{13}{1}$$

possibilities. For red and another color we have

$$\binom{4}{2}\binom{17}{1}$$

possibilities. For blue and another color we have

$$\binom{9}{2}\binom{12}{1}$$

possibilities. All possible choices of three balls is

$$\binom{21}{3}.$$

Thus the probability we seek is

$$\frac{\binom{8}{2}\binom{13}{1}+\binom{4}{2}\binom{17}{1}+\binom{9}{2}\binom{12}{1}}{\binom{21}{3}}.$$

#31 Suppose you have five green balls, two red balls, and three blues balls. Pick two balls with replacement. What is the probability they are the same?

Solution:

They can both be green, both be red, or both be blue. Thus we have

$$\frac{\binom{5}{2}+\binom{2}{2}+\binom{3}{2}}{\binom{10}{2}}.$$

Alternatively, we could have thought of it as

$$\frac{5\cdot 4}{10\cdot 9}+\frac{2\cdot 1}{10\cdot 9}+\frac{3\cdot 2}{10\cdot 9}.$$

#32 Suppose you have nine green balls, seven red balls, and six blues balls. Pick nine balls without replacement. What is the probability of getting three of each color?

Solution:

We have

$$\binom{9}{3}\binom{7}{3}\binom{6}{3}$$

ways of choosing three of each color and a total of

$$\binom{22}{3}$$

ways of choosing three balls. The probability is their quotient.

3.7. Some Facts.

#33 Give the formula for the probability mass function of a Poisson distribution X with parameter $\lambda = 3$? In that case, what $\mathbb{E}(X)$ and $\mathrm{var}(X)$?

Solution:

A random variable X is a Poisson distribution with parameter $\lambda > 0$ if

$$\mathcal{P}(X=k)=e^{-\lambda}\frac{\lambda^k}{k!}, \quad k=0,1,2,\dots .$$

Computations tell us that

$$\mathbb{E}(X)=\lambda \quad \text{and} \quad \mathrm{var}(X)=\lambda.$$

Thus a Poisson distribution with parameter $\lambda = 3$ has probability mass function

$$\mathcal{P}(X=k)=e^{-3}\frac{3^k}{k!}, \quad k=0,1,2,\dots,$$

$$\mathbb{E}(X)=3, \quad \text{and} \quad \mathrm{var}(X)=3.$$

#34 What is the density function of the standard normal distribution?

Solution:

It is

$$f(x)=\frac{1}{\sqrt{2\pi}}e^{-x^2/2}, \quad -\infty < x < \infty.$$

Note that the density function for a normal distribution with mean μ and standard deviation σ is

$$g(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/(2\sigma^2)}, \quad -\infty < x < \infty.$$

#35 State the Poisson Approximation to the Binomial Distribution. Why do we use the approximation? When in practice should we use the Poisson Approximation to approximate the Binomial Distribution? (Just do your best and use your own words.)

Solution:

First, I cite the official statement from your book:

Theorem (Poisson Approximation to the Binomial Distribution)**.** *Suppose S_n is binomial distributed with parameters n and p_n. If $p_n \to 0$ as $n \to \infty$ such that $\lim_{n\to\infty} np_n = \lambda > 0$, then*

$$\lim_{n\to\infty} \mathcal{P}(S_n = k) = e^{-\lambda}\frac{\lambda^k}{k!}$$

In my own words, I would say, the Poisson Approximation is a good approximation to the Binomial Distribution when p is small and n is large. Thus for small p and large n, we have

$$\mathcal{P}(S_n = k) \approx \mathcal{P}(X = k),$$

where S_n is a binomial distribution with parameters n and p, and X is a Poisson distribution with parameter $\lambda = np$. We use the approximation, because it is easier to compute.

At the very end of Section 12.6, your book mentions the following rule of thumb: both the normal and the Poisson distribution approximate binomial distributions well when $n \geq 40$. But one should use the Poisson distribution to approximate when $np \leq 5$ and use the normal distribution to approximate when $np \geq 5$.

#36 Using Chebyshev's Inequality, find the number of times you'd have to toss a coin to determine the probability of flipping a heads within 0.05 of its true value with probability at least 0.95.

Solution:

We are looking for n such that

$$\mathcal{P}\left(\left|\overline{X}_n - p\right| \leq 0.05\right) \geq 0.95 \iff \mathcal{P}\left(\left|\overline{X}_n - p\right| \geq 0.05\right) \leq 1 - 0.95$$

Recalling that $\text{var}(\overline{X}_n) = \frac{p(1-p)}{n}$, we have

$$\begin{aligned}
\mathcal{P}(\left|\overline{X}_n - p\right| \geq 0.05) &\leq \frac{\sigma^2}{(0.05)^2} \\
&= 400\frac{p(1-p)}{n} \\
&\leq 400\left(\frac{1}{2}\right)\left(1 - \frac{1}{2}\right)\frac{1}{n} \\
&= \frac{100}{n}
\end{aligned}$$

and it suffices to choose n so that

$$\frac{100}{n} \leq 1 - 0.95.$$

We conclude that

$$n \geq 2,000$$

#37 State the Central Limit Theorem. (Just do your best and use your own words.)

Solution:

First, I cite the official statement from your book:

Theorem (Central Limit Theorem)**.** *Suppose $X_1, X_2, \ldots, X_n$ are i.i.d. with mean $\mathbb{E}(X_i) = \mu$ and variance $\text{var}(X_i) = \sigma^2 < \infty$. Define $S_n = \sum_{i=1}^{n} X_i$. Then as $n \to \infty$,*

$$\mathcal{P}\left(\frac{S_n - n\mu}{\sqrt{n\sigma^2}} \leq x\right) \to F(x)$$

where $F(x)$ is the distribution function of the standard normal distribution.

In my own words, I would say, for large n, we have

$$\mathcal{P}(S_n \leq k) \approx \mathcal{P}(\mathcal{N} \leq k),$$

where S_n is n repetitions of an experiment X, and $\mathcal{N}$ is a normal distribution with the same mean and standard deviation as S_n. Since X has mean μ and standard deviation σ, S_n has mean $n\mu$ and standard deviation $\sqrt{n}\sigma$.

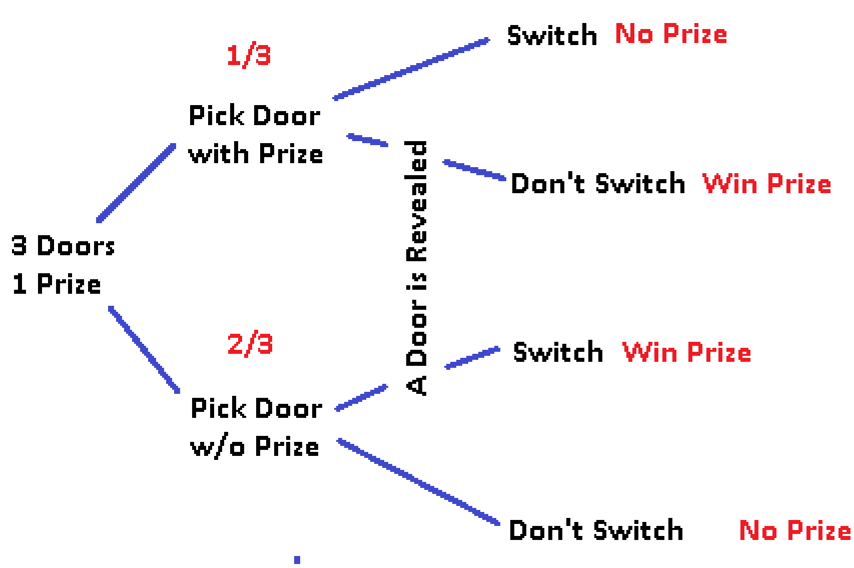
At the very end of Section 12.6, your book mentions the following rule of thumb: both the normal and the Poisson distribution approximate binomial distributions well when $n \geq 40$. But one should use the Poisson distribution to approximate when $np \leq 5$ and use the normal distribution to approximate when $np \geq 5$.

3.8. **Monty Hall.** I present some variations and work out the solutions. There may be better solutions, but these will have to do.

#38 Classic. There are three doors and one prize. You select a door and then Monty reveals a door which doesn't have a prize. What is the probability of getting a prize if you switch? if you don't?

Solution:

We have the following tree:

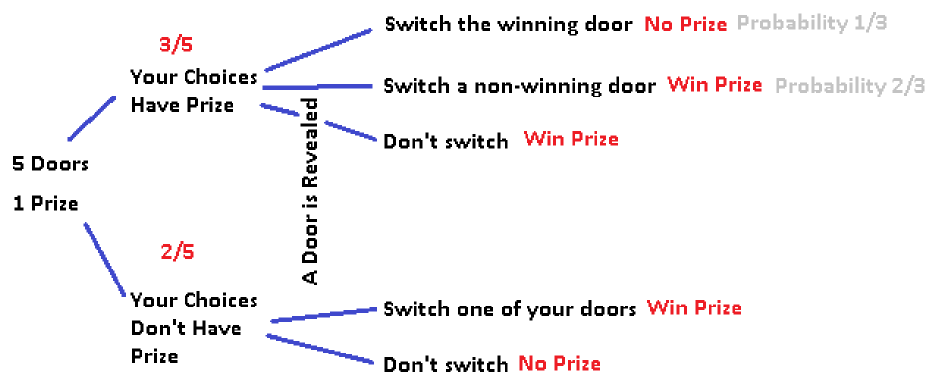


Thus if you switch, you will win a prize with probability $\frac{2}{3}$. If you don't switch, then you will win a prize with probability $\frac{1}{3}$.

#39 There are five doors and one prize. You select three doors and Monty reveals a door which doesn't have a prize. What is the probability of getting a prize if you switch one of your doors? if you don't switch any?

Solution:

We have the following tree:



Thus if you switch, then you will win a prize with probability $\frac{3}{5} \cdot \frac{2}{3} + \frac{2}{5} = \frac{4}{5}$. If you don't switch any, then you will win a prize with probability $\frac{3}{5}$.

#40 There are seven doors and two prizes. You select two doors and Monty reveals a door which doesn't have a prize. What is the expected number of prizes if you switch both doors? just one? no doors?

Solution:

The probability of your first two doors containing both prizes is

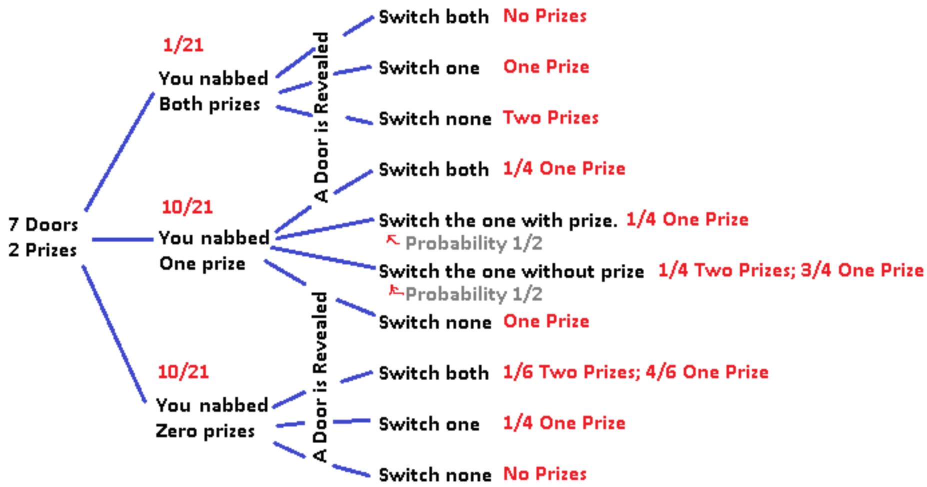
$$\frac{\binom{2}{2}}{\binom{7}{2}} = \frac{1}{21}.$$

The probability of just getting one in your first two doors is

$$\frac{\binom{2}{1}\binom{5}{1}}{\binom{7}{2}} = \frac{10}{21}.$$

Just to be sure, the probability of getting no prizes is

$$\frac{\binom{5}{2}}{\binom{7}{2}} = \frac{10}{21}$$



Switching both results in an expected value of

$$\frac{1}{21}\cdot 0+\frac{10}{21}\cdot\frac{1}{4}\cdot 1+\frac{10}{21}\left(\frac{1}{6}\cdot 2+\frac{4}{6}\cdot 1\right)=\frac{10}{21}\cdot\frac{5}{4}=\frac{12.5}{21}$$

prizes, switching one results in an expected value of

$$\frac{1}{21}\cdot 1+\frac{10}{21}\frac{1}{2}\frac{1}{4}\cdot 1+\frac{10}{21}\frac{1}{2}\left(\frac{1}{4}\cdot 2+\frac{3}{4}\cdot 1\right)+\frac{10}{21}\frac{1}{4}\cdot 1=\frac{11}{21}$$

prizes, and switching none results in an expected value of

$$\frac{1}{21}\cdot 2+\frac{10}{21}\cdot 1+\frac{10}{21}\cdot 0=\frac{12}{21}$$

prizes. As a result, we find switching both is slightly better than switching none, which is slightly better than switching one.

#41 There are ten doors and three prizes. You select three doors and Monty reveals a door which doesn't have a prize. What is the expected number of prizes if you switch all three doors?

Solution:

Because we're only concerned with switching all three doors, we no longer need to draw a diagram.

Case 1: Choosing three doors with the prizes behind them has a probability of $\frac{1}{\binom{10}{3}}$. Switching all three doors guarantees getting no prizes in this case.

Case 2: Choosing three doors of which there are two prizes behind two of them has a probability $\frac{\binom{3}{2}\binom{7}{1}}{\binom{10}{3}}$. Switching all three doors leaves us with choosing three doors among six doors and one prize. The probability of getting that prize is $\frac{3}{6}=\frac{1}{2}$.

Case 3: Choosing three doors of which there is one prize behind one of them has probability $\frac{\binom{3}{1}\binom{7}{2}}{\binom{10}{3}}$. Switching all three doors leaves us with choosing three doors among six door and two prizes. The probability of getting both prizes is $\frac{1}{\binom{6}{3}}$. The probability of getting one prize is $\frac{\binom{2}{1}\binom{4}{1}}{\binom{6}{3}}$.

Case 4: Choosing three doors of which there are no prizes behind any of them has probability $\frac{\binom{7}{3}}{\binom{10}{3}}$. Switching all three doors leaves us with choosing three doors among six doors and three prizes. To get three prizes we have $\frac{1}{\binom{6}{3}}$, two prizes is $\frac{\binom{3}{2}\binom{3}{1}}{\binom{6}{3}}$, and one prize is $\frac{\binom{3}{1}\binom{3}{2}}{\binom{6}{3}}$.

Thus we have

$$\frac{\binom{3}{2}\binom{7}{1}}{\binom{10}{3}}\frac{1}{2}\cdot 1+\frac{\binom{3}{1}\binom{7}{2}}{\binom{10}{3}}\left(\frac{1}{\binom{6}{3}}\cdot 2+\frac{\binom{2}{1}\binom{4}{1}}{\binom{6}{3}}\cdot 1\right)+\frac{\binom{7}{3}}{\binom{10}{3}}\left(\frac{1}{\binom{6}{3}}\cdot 3+\frac{\binom{3}{2}\binom{3}{1}}{\binom{6}{3}}\cdot 2+\frac{\binom{3}{1}\binom{3}{2}}{\binom{6}{3}}\cdot 1\right),$$

which simplifies to

$$\frac{189}{240}.$$

*Provided I did my arithmetic correctly.

3.9. **Hemophilia Pedigree.** #42, #43, #44, #45 I will not write four problems on this. Please refer to your book for example problems on this topic.

3.10. **Probability Distributions.** #46, #47, #48 (Version 1) Suppose you are given a coin with probability $\frac{1}{10}$ of showing up heads. After some time, you have flipped the coin 50 times and have recorded the number of times it came up heads. (a) What is the expected number of times it should come up heads? (b) What is the probability that it comes up heads at least two times? Use the fact that

$$\left(\frac{9}{10}\right)^{49} \approx 0.005726$$

in your computation of (b). (c) Use the Poisson distribution to approximate the same probability. Use the fact that

$$e^{-5} \approx 0.006738$$

in your computation of (c).

Solution:

Let S_{50} denote the binomial distribution of flipping a coin of probability $\frac{1}{10}$ a total of 50 times.

(a) The expectation that the coin should come up heads is $\mathbb{E}(S_{50}) = np = 50 \cdot \frac{1}{10} = 5$ times.

(b) The complement of this is the probability it comes up heads exactly 0 times and exactly 1 time. We have

$$\begin{aligned}
\mathcal{P}(S_{50} \geq 2) &= 1 - (\mathcal{P}(S_{50} = 0) + \mathcal{P}(S_{50} = 1)) \\
\mathcal{P}(S_{50} = 0) + \mathcal{P}(S_{50} = 1) &= \binom{50}{0}\left(\frac{1}{10}\right)^{0}\left(\frac{9}{10}\right)^{50} + \binom{50}{1}\left(\frac{1}{5}\right)^{1}\left(\frac{9}{10}\right)^{49} \\
&\approx \frac{9}{10}(0.005726) + 10(0.005726) \\
&= 9 \cdot 0.0005726 + 0.05726 \\
&= 0.0051534 + 0.0572600 \\
&= 0.0624134.
\end{aligned}$$

We conclude

$$\mathcal{P}(S_{50} \geq 2) \approx 1 - 0.0624134 = 0.9375866$$

(c) By the Poisson Approximation to the Binomial Distribution, we have

$$\mathcal{P}(S_{50} = k) \approx \mathcal{P}(X = k)$$

where X is a Poisson distribution with parameter $\lambda = np = 5$. In this case we have

$$\begin{aligned}\mathcal{P}(S_{50} = 0) + \mathcal{P}(S_{50} = 1) &\approx \mathcal{P}(X = 0) + \mathcal{P}(X = 1)\\ &= e^{-5}\frac{5^0}{0!} + e^{-5}\frac{5^1}{1!}\\ &= 0.006738 \cdot (1 + 5)\\ &= 6 \cdot 0.006738\\ &= 0.040428.\end{aligned}$$

We conclude

$$\mathcal{P}(S_{50} \geq 2) \approx 1 - 0.040428 = 0.959572.$$

#46, #47, #48 (Version 2) Suppose you are given a coin with probability $\frac{1}{8}$ of showing up heads. After some time, you have flipped the coin 448 times and have recorded the number of times it came up heads. (a) What is the expected number of times it should come up heads? (b) Using the normal approximation (no histogram adjustment), what is the probability that it comes up heads at least 49 times? If applicable, use the empircal rule. (c) Using the Poisson distribution, what is the probability that it comes up heads at least 49 times? Use the fact that

$$\sum_{k=0}^{48} \frac{56^k}{k!} \approx 3.30 \times 10^{23}$$

and

$$e^{-56} \approx 4.78 \times 10^{-25}$$

Let S_{448} denote the binomial distribution of flipping a coin of probability $\frac{1}{8}$ a total of 448 times.

(a) The expectation that the coin should come up heads is $\mathbb{E}(S_{448}) = np = 448 \cdot \frac{1}{8} = 56$.

(b) By the central limit theorem

$$\mathcal{P}(S_{448} \geq 49) \approx \mathcal{P}(\mathcal{N} \geq 49)$$

where $\mathcal{N}$ is the normal distribution with mean $np = 56$ and standard deviation $\sqrt{np(1-p)} = \sqrt{56 \cdot \frac{7}{8}} = 7$. Then we have

$$\begin{aligned}\mathcal{P}(\mathcal{N} \geq 49) &= \mathcal{P}(\mathcal{N} - 56 \geq -7)\\ &= \mathcal{P}\left(\frac{\mathcal{N} - 56}{7} \geq -1\right)\\ &= \mathcal{P}(Z \geq -1)\end{aligned}$$

where Z is the standard normal distribution.

Remark. The key here is to note we are looking for the area beyond one standard deviation under the mean. If we already noticed this with $\mathcal{N}$, then we don't need to normalize to the standard normal distribution Z.

Using the $68 - 95 - 99$ rule (also known as the emprical rule), we have

$$\mathcal{P}(Z \geq -1) \approx 0.34 + 0.5 = 0.84.$$

Remark. Note that 0.68 is the area within one standard deviation. Thus the region between $-1 \leq Z \leq 0$ has area approximately 0.34.

(c) By the Poisson Approximation to the Binomial Distribution, we have

$$\mathcal{P}(S_{448} = k) \approx \mathcal{P}(X = k)$$

where X is a Poisson distribution with parameter $\lambda = np = 56$. Then

$$\begin{aligned}\mathcal{P}(S_{448} \leq 48) &\approx \mathcal{P}(X \leq 48) \\ &= \sum_{k=0}^{48} e^{-\lambda}\frac{\lambda^k}{k!} \\ &\approx (4.78 \times 10^{-25})(3.30 \times 10^{23}) \\ &= 15.774 \times 10^{-2} \\ &= 0.15774.\end{aligned}$$

Thus

$$\mathcal{P}(S_{448} \geq 49) \approx 0.84226$$

4. Change Log

v1.1.0.0
Added #46, #47, #48 (Version 1)
Added #46, #47, #49 (Version 2)