PRACTICE EXAM

1. Midterm 1 Material

1.1. **Differential Equation.**

1.1.1. *#1.* Solve the following differential equation when $x = \sqrt{\frac{3}{5}}$ and $y = 1$.

$$\frac{dy}{dx} = -4y^4x$$

1.1.2. *#2.* Find the equilibrium points for the differential equation.

$$\frac{dy}{dx} = (y+1)(y-1)(y-2)$$

1.1.3. *#3.* Which equilibrium points are stable and which are unstable?

1.2. **Leslie Matrices.**

1.2.1. *#4.* We have a population of newborns, N_0, and one-year olds, N_1. There are no two-year olds or older. One third of the newborns survive to the next year to be one-year olds. Each newborn produces 1 newborn for the next year.. Each one-year old produces 6 newborns for the next year. What is the Leslie matrix that takes

$$\begin{bmatrix} N_0(t) \\ N_1(t) \end{bmatrix} \quad \text{to} \quad \begin{bmatrix} N_0(t+1) \\ N_1(t+1) \end{bmatrix}?$$

1.2.2. *#5.* Find the eigenvalues.

1.2.3. *#6.* Find the associated eigenvectors.

1.2.4. *#7.* What is a stable age distribution $\binom{a}{b}$ for this population?

2. Midterm 2 Material

2.1. **System of Differential Equations.** #8,#9 Solve the system of differential equations:

$$x'(t) = 2x(t) + y(t)$$
$$y'(t) = 7x(t) - 4y(t)$$

Find the eigenvalues, Find the eigenvectors.

#10 Suppose the previous question had the initial conditions

$$\begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} 5 \\ -3 \end{bmatrix}.$$

Solve the initial value problem. (That is, solve for the c_1 and c_2)

#11 What type of equilibrium point is at $(0,0)$?

2.2. **Questions that are Asked of a Function with Two Variables.** That's right... TWO!

$$f(x, y) = 4x^2 + 2x + y^2$$

#12 Compute the gradient of f, i.e. ∇f.

#13 Compute the gradient at the point $(1, 2)$. Same question: Compute the direction of maximum slope at $(1, 2)$.

#14 Compute the slope of f in the direction $(-3, 4)$ at the point $(1, 2)$.

#15 Compute the slope of f at $(1, 2)$ in the direction it is maximal.

$$f(x, y) = 4x^2 + 2x + y^2$$

#16 Find the equation for the tangent plane to f at the point $(1, 2)$. Place the solution in the form

$$z - z_0 = a(x - x_0) + b(y - y_0).$$

#17 Give a parametric equation for the tangent line to the graph of f (in 3-space) for $(x, y) = (1, 2)$ in the direction $(-3, 4)$.

#18 Compute the Hessian of f.

#19 Find the one critical point for f.

$$f(x, y) = 4x^2 + 2x + y^2$$

#20, #21, #22, #23 Consider the function on the domain $x^2 + y^2 \leq 1$. What is the minimum value f takes and where? What is the maximum f takes and where?

3. Newest Material - Probability and Statistics

3.1. **Sandwich.** #24 Suppose you have three types of bread (white, wheat, and pita), two types of protein (mushroom and turkey), two types of extras (pickle and tomato), and three types of cheese (provolone, American, and cheddar). Using one type of bread on either end and concerned about the order of ingredients, how many sandwiches can you make?

3.2. **Counting.** #25 Suppose there are nine females and six males at a dance. How many ways are there to form four couples on the dance floor?

3.3. **Standard Deck of Cards.** #26a What's the probability of getting a full house when picking five cards? A full house is a three of a kind and a pair.

Try your "hand" at other Poker hands.
#26b Royal Flush (10 J Q K A of the same suit).

#26c Straight Flush (five consecutive cards of the same suit). This includes the straight starting with an ace-low, that is A2345. For simplicity, we'll include the Royal Flush.

#26d Four of a kind

#26e Flush (five cards of the same suit). For simplicity, we'll include Royal Flushes and Straight Flushes.

#26f Straight (five consecutive cards). This includes the straight starting with an ace-low, that is A2345. For simplicity, we'll include the Royal Flush.

#26g Skipping three of a kind, we consider two pair and exclude the case of three of a kind and four of a kind. Thus the two pairs are distinct and there's a fifth card different from the values of the two pairs.

3.4. **Sample Space.** #27 Determine the sample space for flipping a coin three times. What is the probability of getting at least two heads?

3.5. **Calculating Expected Value.** #28 Suppose you meet someone on the street telling you that he'll roll a die and if it comes up odd you have to pay a two dollars, but it if it comes up a two or four you will win a quarter and if it comes up a six you will win five dollars. If you play, what is your expected value?

3.6. **Colored Balls.** #29 Suppose you have four green balls, five red balls, and three blues balls. Pick seven balls with replacement (take, note the color, replace, repeat five times). What is the probability of having the number of red balls be greater than the number of blue balls by one?

#30 Suppose you have eight green balls, four red balls, and nine blues balls. Pick three balls without replacement. What is the probability you get three balls from two different color groups?

#31 Suppose you have five green balls, two red balls, and three blues balls. Pick two balls with replacement. What is the probability they are the same?

#32 Suppose you have nine green balls, seven red balls, and six blues balls. Pick nine balls without replacement. What is the probability of getting three of each color?

3.7. **Some Facts.** #33 Give the formula for the probability mass function of a Poisson distribution X with parameter $\lambda = 3$? In that case, what $\mathbb{E}(X)$ and $\mathrm{var}(X)$?

#34 What is the density function of the standard normal distribution?

#35 State the Poisson Approximation to the Binomial Distribution. Why do we use the approximation? When in practice should we use the Poisson Approximation to approximate the Binomial Distribution? (Just do your best and use your own words.)

#36 Using Chebyshev's Inequality, find the number of times you'd have to toss a coin to determine the probability of flipping a heads within 0.05 of its true value with probability at least 0.95.

#37 State the Central Limit Theorem. (Just do your best and use your own words.)

3.8. **Monty Hall.** I present some variations and work out the solutions. There may be better solutions, but these will have to do.

#38 Classic. There are three doors and one prize. You select a door and then Monty reveals a door which doesn't have a prize. What is the probability of getting a prize if you switch? if you don't?

#39 There are five doors and one prize. You select three doors and Monty reveals a door which doesn't have a prize. What is the probability of getting a prize if you switch one of your doors? if you don't switch any?

#40 There are seven doors and two prizes. You select two doors and Monty reveals a door which doesn't have a prize. What is the expected number of prizes if you switch both doors? just one? no doors?

#41 There are ten doors and three prizes. You select three doors and Monty reveals a door which doesn't have a prize. What is the expected number of prizes if you switch all three doors?

3.9. **Hemophilia Pedigree.** #42, #43, #44, #45 I will not write four problems on this. Please refer to your book for example problems on this topic.

3.10. **Probability Distributions.** #46, #47, #48 (Version 1) Suppose you are given a coin with probability $\frac{1}{10}$ of showing up heads. After some time, you have flipped the coin 50 times and have recorded the number of times it came up heads. (a) What is the expected number of times it should come up heads? (b) What is the probability that it comes up heads at least two times? Use the fact that

$$\left(\frac{9}{10}\right)^{49} \approx 0.005726$$

in your computation of (b). (c) Use the Poisson distribution to approximate the same probability. Use the fact that

$$e^{-5} \approx 0.006738$$

in your computation of (c).

#46, #47, #48 (Version 2) Suppose you are given a coin with probability $\frac{1}{8}$ of showing up heads. After some time, you have flipped the coin 448 times and have recorded the number of times it came up heads. (a) What is the expected number of times it should come up heads? (b) Using the normal approximation (no histogram adjustment), what is the probability that it comes up heads at least 49 times? If applicable, use the empircal rule. (c) Using the Poisson distribution, what is the probability that it comes up heads at least 49 times? Use the fact that

$$\sum_{k=0}^{48} \frac{56^k}{k!} \approx 3.30 \times 10^{23}$$

and

$$e^{-56} \approx 4.78 \times 10^{-25}$$

4. Change Log

v1.1.0.0
Added #46, #47, #48 (Version 1)
Added #46, #47, #49 (Version 2)