1) Compute the eigenvalues λ_1 and λ_2 and corresponding eigenvectors for the matrix

$$A = \begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix}.$$

Solution: To compute the eigenvalues we need to solve the eigenvalue equation

$$0 = \det \begin{bmatrix} 2-\lambda & 1 \\ 2 & 3-\lambda \end{bmatrix} = (\lambda-2)(\lambda-3)-2 = \lambda^2-5\lambda+4 = (\lambda-1)(\lambda-4).$$

Thus, the eigenvalues are $\lambda_1 = 1$, and $\lambda_2 = 4$.

The eigenvectors for $\lambda_1 = 1$ will be solutions of

$$\begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = 1 \cdot \begin{bmatrix} u_1 \\ u_2 \end{bmatrix},$$

which can be rewritten as

$$\begin{aligned} 2u_1 + u_2 &= u_1 \\ 2u_1 + 3u_2 &= u_2, \end{aligned}$$

which reduces to $u_1 + u_2 = 0$. Thus the eigenvectors for $\lambda_1 = 1$ are $\{(t, -t) : t \in \mathbb{R}\}$.

The eigenvectors for $\lambda_2 = 4$ will be solutions of

$$\begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = 4 \cdot \begin{bmatrix} u_1 \\ u_2 \end{bmatrix},$$

which can be rewritten as

$$\begin{aligned} 2u_1 + u_2 &= 4u_1 \\ 2u_1 + 3u_2 &= 4u_2, \end{aligned}$$

which reduces to $2u_1 - u_2 = 0$. Thus the eigenvectors for $\lambda_2 = 4$ are $\{(t, 2t) : t \in \mathbb{R}\}$.

2) If A is the matrix in the previous problem, compute

$$A^4 \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

Solution: The vector $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is an eigenvector of A with eigenvalue 4. Therefore, since $4^4 = 256$

$$A^4 \begin{bmatrix} 1 \\ 2 \end{bmatrix} = 4^4 \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 256 \\ 512 \end{bmatrix}.$$

3) Find the linear approximation to

$$f(x, y) = \begin{bmatrix} ye^{-x} \\ \sin x + \cos y \end{bmatrix}$$

at $(0, 0)$.

Solution: We first compute the Jacobi matrix of f:

$$\begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{bmatrix} = \begin{bmatrix} -ye^{-x} & e^{-x} \\ \cos x & -\sin y \end{bmatrix}.$$

This matrix evaluated at $(0, 0)$ is

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Therefore, the linear approximation to f at $(0, 0)$ will be

$$L(x, y) = f(0, 0) + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} y \\ x \end{bmatrix}$$
$$= \begin{bmatrix} y \\ 1 + x \end{bmatrix}.$$

4) In which direction does

$$f(x, y) = 3xy - \frac{1}{2}x^2$$

increase most rapidly at $(1, 1)$?

Solution: The gradient of f is

$$\nabla f(x, y) = \begin{bmatrix} 3y - x \\ 3x \end{bmatrix}.$$

Since,

$$\nabla f(1, 1) = \begin{bmatrix} 2 \\ 3 \end{bmatrix},$$

and $|\nabla f(1,1)| = \sqrt{2^2+3^2} = \sqrt{13}$, the function increases most rapidly in the direction

$$\begin{bmatrix} 2/\sqrt{13} \\ 3/\sqrt{13} \end{bmatrix}.$$

5) Let

$$A = \begin{bmatrix} 1 & 3 \\ -1 & -2 \end{bmatrix}, \quad \text{and } x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}.$$

Find the stable equilibria (if any) of the equation

$$\frac{d}{dt}x(t) = Ax(t).$$

Solution: Let us first compute the eigenvectors. These are solutions of the eigenvalue equation

$$0 = \det \begin{bmatrix} 1-\lambda & 3 \\ -1 & -2-\lambda \end{bmatrix} = (\lambda - 1)(\lambda + 2) + 3 = \lambda^2 + \lambda + 1.$$

The solutions are given by the quadratic formula

$$\lambda_{1,2} = \frac{-1 \pm \sqrt{1-4}}{2} = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}.$$

Since the eigenvalues are complex conjugates with negative real part, we conclude that $(0,0)$ is the only equilibrium. Indeed, any solution, $x(t)$, of the system will tend to $(0,0)$ as $t \to +\infty$.

6) Consider the system with

$$\frac{dx_1}{dt} = -2x_1 + x_2$$
$$\frac{dx_2}{dt} = 2x_1 - x_2,$$

with the initial condition $x_1(0) = 4$, $x_2(0) = 5$. Find $\lim_{t\to+\infty} x_1(t)$, and $\lim_{t\to+\infty} x_2(t)$.

Solution: We first compute the eigenvalues of the system. These are solutions of the eigenvalue equation

$$0 = \begin{bmatrix} -2-\lambda & 1 \\ 2 & -1-\lambda \end{bmatrix} = (\lambda + 2)(\lambda + 1) - 2 = \lambda^2 + 3\lambda.$$

Thus, the two eigenvalues are $\lambda_1 = 0$, and $\lambda_2 = -3$.

We next find eigenvectors. The ones for $\lambda_1 = 0$ are solutions of the equation

$$\begin{aligned} -2u_1 + u_2 &= 0 \\ 2u_1 - u_2 &= 0, \end{aligned}$$

which reduces to $u_2 = 2u_1$. Thus, $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is an eigenvector for the eigenvalue $\lambda_1 = 0$. The eigenvectors for $\lambda_2 = -3$ are solutions of

$$\begin{aligned} -2u_1 + u_2 &= -3u_1 \\ 2u_1 - u_2 &= -3u_2, \end{aligned}$$

which reduces to $u_1 + u_2 = 0$. Thus, $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is an eigenvector for $\lambda_2 = -3$.

Because of these calculations we know that the solution must be of the form

$$c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 e^{-3t} \begin{bmatrix} 1 \\ -1 \end{bmatrix},$$

where the constants c_1 and c_2 are determined by the initial conditions:

$$\begin{bmatrix} 4 \\ 5 \end{bmatrix} = \begin{bmatrix} c_1 + c_2 \\ 2c_1 - c_2 \end{bmatrix}.$$

By adding the two rows, we find that $c_1 = 3$, which in turn leads to $c_2 = 1$. Therefore, the solution is

$$x(t) = 3 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + e^{-3t} \begin{bmatrix} 1 \\ -1 \end{bmatrix},$$

which yields the answer

$$\lim_{t \to +\infty} x_1(t) = 3, \quad \text{and} \quad \lim_{t \to +\infty} x_2(t) = 6.$$