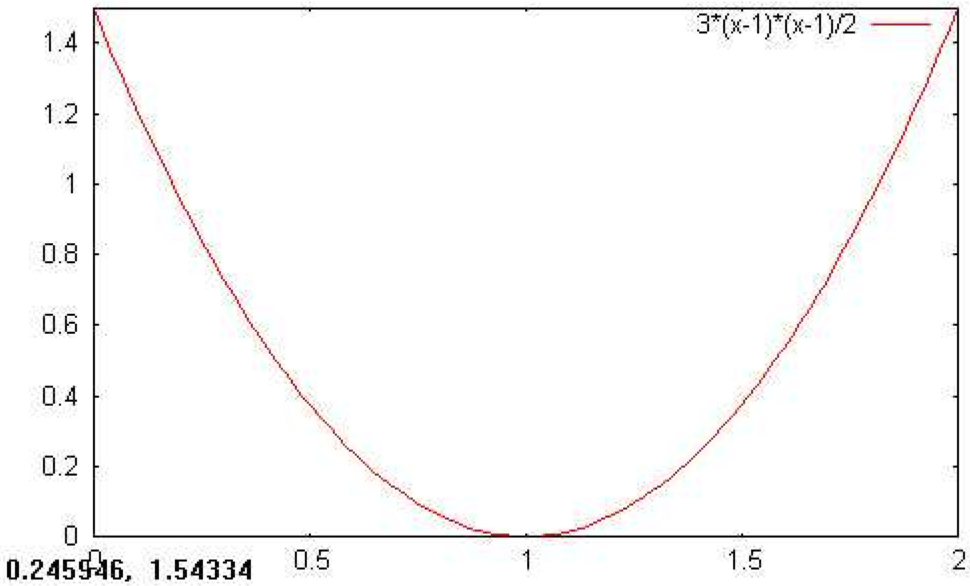
Math 107, Spring 2006: Final Exam Practice Question Solutions

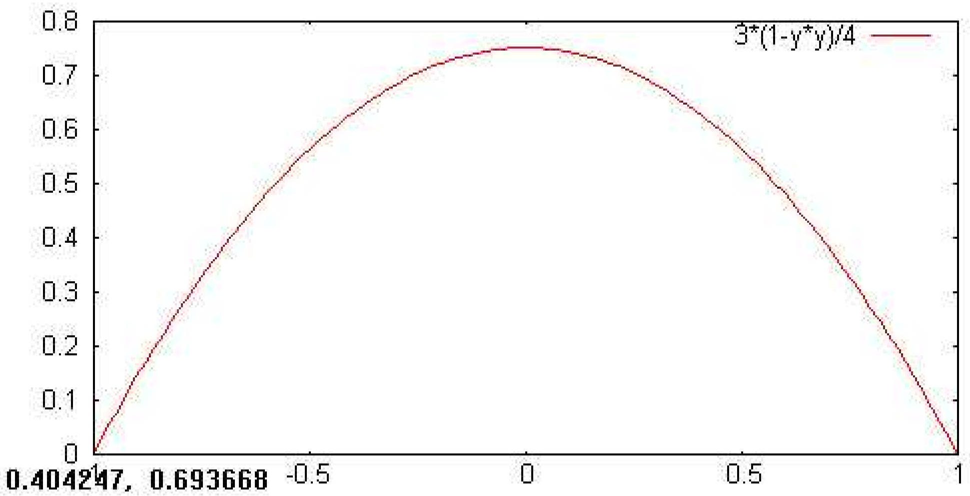
Please let me know if you find any errors, or if you don't understand any of the solutions.

Solutions

1. (a) The graph of the pdf of X is:



The graph of the pdf of Y is:



(Apologies for the weird numbers that appear in the bottom-left corners of the graphs.)

(b) From the symmetry of the graphs you would expect the expectation of X to be 1 (that is, in the middle of the graph. Similarly, the expectation of Y should be 0. Therefore, the expectation of X is larger.

(c) The range of possible values for each of X and Y is 1 unit either side of the expectation. But for X, it is the values further from the expectation that are more likely, whereas for Y, it is the values closer to the expectation that are more likely. Therefore, you would expect the variance of X to be larger than the variance of Y.

2. For a discrete random variable, the expectation is given by the following formula:

$$EX = \sum_x x\, P(X = x) = 0 \times 0.2 + 0.5 \times 0.4 + 1 \times 0.1 + 1.5 \times 0.2 + 2 \times 0.1 = 0.8$$

The variance is then

$$\begin{aligned} \operatorname{Var} X &= \sum_x (x - EX)^2\, P(X = x) \\ &= (0 - 0.8)^2 \times 0.2 + (0.5 - 0.8)^2 \times 0.4 + (1 - 0.8)^2 \times 0.1 \\ &\quad + (1.5 - 0.8)^2 \times 0.2 + (2 - 0.8)^2 \times 0.1 \\ &= 0.128 + 0.036 + 0.004 + 0.098 + 0.144 \\ &= 0.41. \end{aligned}$$

(The calculations in this question were intended to be a little easier, but I messed up the numbers. This is harder than anything you will need to do on the exam. But it is good practice because you will have to do some calculations of this sort.)

3. If X has uniform distribution on the interval $[1, 3]$, then its pdf is equal to $1/2$ for $1 \le x \le 3$ and 0 otherwise. Therefore

$$EX = \int_1^3 x\, \frac{1}{2}\, dx = [x^2/4]_1^3 = (9 - 1)/4 = 2$$

and

$$\operatorname{Var} X = \int_1^3 (x - EX)^2\, \frac{1}{2}\, dx = \int_1^3 \frac{(x-2)^2}{2}\, dx = [(x-2)^3/6]_1^3 = 1/6 - (-1/6) = 1/3.$$

4. (a) We want to find $P(X_{63} \ge 65)$. Since X_{63} is normally distributed with mean 63 and standard deviation 2, we set

$$Z = \frac{X_{63} - 63}{2}$$

which is then a standard normal distribution. Then

$$P(X_{63} \ge 65) = P(Z \ge (65 - 63)/2) = P(Z \ge 1) = 1 - 0.8413 = 0.1587.$$

(b) The probability that the gun reads more than 65 on one occasion is equal to $1 - P(\text{reads less than 65 on both occasions})$. Assuming that the two readings are independent, this is just

$$1 - P(\text{reads less than 65 first time})\; P(\text{reads less than 65 second time})$$

which is equal to:

$$1 - P(X_{67.56} \leq 65)P(X_{69} \leq 65).$$

As in part (a), we find these probabilities by converting to standard normal distributions:

$$P(X_{67.56} \leq 65) = P(Z \leq (65 - 67.56)/2) = P(Z \leq -1.28) = 1 - 0.9 = 0.1$$

and

$$P(X_{69} \leq 65) = P(Z \leq (65 - 69)/2) = P(Z \leq -2) = 1 - 0.9772 = 0.0228.$$

Therefore, the probability we want is

$$1 - 0.1 \times 0.0228 = 1 - 0.00228 = 0.99772.$$

5. (a) The standard error for these observations is

$$\frac{\text{sample standard deviation}}{\sqrt{n}} = \frac{10}{\sqrt{100}} = 1.$$

Therefore, this confidence interval is of the form

$$\text{sample mean} \pm 2 \times \text{standard error}$$

The confidence level is therefore equal to $P(-2 \leq Z \leq 2)$ where Z is a standard normal distribution. This is equal to

$$P(Z \leq 2) - P(Z \leq -2) = 0.9772 - (1 - 0.9772) = 0.9772 - 0.0228 = 0.9544.$$

So this is a 95.44% confidence interval.

(b) To find an 80% confidence interval, we need to find z such that

$$P(-z \leq Z \leq z) = 0.8.$$

But $P(-z \leq Z \leq z) = P(Z \leq z) - (1 - P(Z \leq z)) = 2P(Z \leq z) - 1$. Therefore we want

$$2P(Z \leq z) - 1 = 0.8$$

or $P(Z \leq z) = 0.9$. This is true for $z = 1.28$. Therefore the confidence interval we want is of the form

$$\text{sample mean} \pm 1.28 \times \text{standard error} = 24 \pm 1.28 \times 1 = [22.72, 25.28].$$

6. (a) i. We could work out the probabilities of all the different scores and use the usual expectation formula for a discrete random variable, but here is a quicker way. Let X_1 be the score on the first die and X_2 the score on the second die. Then $X = X_1 + X_2$. The expectation of X_1 is

$$1 \times \frac{1}{6} + 2 \times \frac{1}{6} + \cdots + 6 \times \frac{1}{6} = 3.5.$$

Similarly, the expectation of X_2 is 3.5. Therefore:

$$EX = EX_1 + EX_2 = 3.5 + 3.5 = 7.$$

ii. We can do a similar trick with Y as we did with X. If Y_1 is the random variable that is 1 is the first roll is a 6 and 0 otherwise, and Y_2 is the same thing for the second roll, then $Y = Y_1 + Y_2$. The expectation of each of Y_1 and Y_2 is

$$0 \times \frac{5}{6} + 1 \times \frac{1}{6} = \frac{1}{6}.$$

Therefore, the expectation of Y is

$$EY = 1/6 + 1/6 = 1/3.$$

iii. For XY, we have to work out the full distribution. But note that if we don't roll any six then $XY = 0$. The other possible values for XY are

outcomes	value of XY	probability of that value
16, 61	7	2/36
26, 62	8	2/36
36, 63	9	2/36
46, 64	10	2/36
56, 65	11	2/36
66	24	1/36

Therefore the expectation of XY is

$$7 \times \frac{2}{36} + 8 \times \frac{2}{36} + 9 \times \frac{2}{36} + 10 \times \frac{2}{36} + 11 \times \frac{2}{36} + 24 \times \frac{1}{36} = \frac{19}{6}.$$

(b) If X and Y were independent, we would have $E(XY) = EX.EY$, but that is not true here because

$$\frac{19}{6} \neq 7 \times \frac{1}{3}.$$

We would not have expected X and Y to be independent, because one of the random variables does affect the other. Having large numbers of sixes makes it more likely to get a large total score than you would get otherwise.

7. (a) HHH, HHT, HTH, HTT, THH, THT, TTH, TTT

(b) i. outcomes: HHH, TTT; probability: $1/8 + 1/8 = 1/4$.

ii. outcomes: HHT, HTT, THH, TTH; probability: $1/8+1/8+1/8+1/8 = 1/2$.

iii. outcomes: HTT, THT, TTH; probability: $1/8 + 1/8 + 1/8 = 3/8$.

(c) The number of heads is binomially distributed with $n = 3$ and $p = 1/2$. Therefore the probability that the number of heads is equal to 1 is:

$$\binom{3}{1} (1/2)^1 (1/2)^2 = 3(1/2)(1/4) = 3/8.$$

8. If $\overline{X}$ is the average of 25 observations of X, then the Central Limit Theorem says that $\overline{X}$ is approximately normally distributed with mean μ and variance σ^2/n where μ and

σ^2 are the expectation and variance, respectively, of X. So the mean of $\overline{X}$ is 11.6 and the variance of $\overline{X}$ is 16/25. Therefore

$$Z = \frac{\overline{X} - 11.6}{\sqrt{16/25}} = \frac{\overline{X} - 11.6}{0.8}$$

is a standard normal distribution and so

$$P(\overline{X} \leq 10) = P(Z \leq (10 - 11.6)/0.8) = P(Z \leq -2) = 1 - 0.9772 = 0.0228.$$

9. The Central Limit Theorem tells us that the number of heads is approximately normally distributed. We can get the mean and variance from the binomially distribution formula with $n = 400$ and $p = 0.2$ So the mean number of heads is $np = 80$ and the variance is $np(1 - p) = 64$. Then

$$Z = \frac{\#\text{ heads} - 80}{\sqrt{64}} = \frac{\#\text{ heads} - 80}{8}$$

is a standard normal distribution. So

$$P(\#\text{ heads} \geq 88) = P(Z \geq (88 - 80)/8) = P(Z \geq 1) = 1 - 0.8413 = 0.1587.$$