Math 107: Calculus II, Spring 2006: Practice Final Solutions

Solutions

1. (a) The equilibrium solution is at $y = 0$. The phase line should have arrows pointing away from $y = 0$.

   (b) Your solution sketch should have a horizontal line at $y = 0$ (the equilibrium solution). For $y > 0$, the solutions should be increasing (i.e. positive slope) and for $y < 0$, they should be decreasing (negative slope).

   (c) Unstable.

   (d) $y = \pm\sqrt{\dfrac{-1}{2t+c}}$ or $y = 0$.

   (e) $y = \sqrt{\dfrac{-1}{2t-16}}$.

2. Putting this system in matrix form we get

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}.$$

To solve this we have to find the eigenvalues and eigenvectors of the matrix

$$\begin{pmatrix} -2 & 1 \\ 0 & 1 \end{pmatrix}.$$

To find eigenvalues, we take the determinant of

$$\begin{pmatrix} -2-k & 1 \\ 0 & 1-k \end{pmatrix}$$

which is equal to

$$(-2-k)(1-k) - 0 = (-2-k)(1-k).$$

This is zero when $k = 1$ or $k = -2$, so these are the two eigenvalues. To find the eigenvectors we solve the equation

$$\begin{pmatrix} -2-k & 1 \\ 0 & 1-k \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}\begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

We get eigenvector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ corresponding to eigenvalue $k = -2$ and eigenvector $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$ corresponding to $k = 1$. Therefore the general solution to the given system of equations is

$$\begin{pmatrix} x \\ y \end{pmatrix} = c_1 e^{-2t}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 e^{t}\begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

and so $x(t) = c_1 e^{-2t} + c_2 e^{t}$ and $y(t) = 3c_2 e^{t}$.

3. (a) The determinant of the given matrix is $2a + 4$. The matrix does not have an inverse when this is equal to zero, i.e. for $a = -2$ only.

   (b) i. When $a = 1$, the matrix does have an inverse, and its inverse is

$$\frac{1}{6}\begin{pmatrix} 2 & -4 \\ 1 & 1 \end{pmatrix}$$

So the solution is

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{6}\begin{pmatrix} 2 & -4 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 10/6 \\ -1/6 \end{pmatrix}.$$

   ii. When $a = -2$, the matrix does not have an inverse. In this case, the system of equations becomes

$$\begin{aligned} -2x + 4y &= 1 \\ -x + 2y &= -2 \end{aligned}$$

Multiplying the second equation by 2, we see that these are inconsistent. Therefore, this system of equations has no solutions.

4. (a) $\partial f/\partial x = (1-y)e^{x+y-xy}$ and $\partial f/\partial y = (1-x)e^{x+y-xy}$.

   (b) $\nabla f(1,0) = (e, 0)$

   (c) $e/\sqrt{2}$

   (d) $L(x,y) = e + e(x-1) + 0(y-0) = e + e(x-1)$

5. (a) The c-level curve is the curve where $y + x^2 = c$ or $y = -x^2 + c$. This is an inverted parabola passing through the point $(0, c)$.

   (b) The $y = 1$ cross-section is a graph of z against x given by $z = 1 + x^2$.

   (c) The gradient vector is $(2x, 1)$ which at $(1,1)$ is $(2,1)$. This vector is perpendicular to the 2-level curve at the point $(1,1)$.

6. (a) The partial derivatives of f are: $\partial f/\partial x = 2xy - 4y$ and $\partial f/\partial y = x^2 - 4x + 2y$. Substituting each of the three points in gives zero for both of these, so they are critical points.

   (b) The Hessian for this function is

$$Hf = \begin{pmatrix} 2y & 2x-4 \\ 2x-4 & 2 \end{pmatrix}.$$

At $(0,0)$ this is

$$\begin{pmatrix} 0 & -4 \\ -4 & 2 \end{pmatrix}$$

which has determinant -16. Therefore $(0,0)$ is a saddle point.

At $(4, 0)$ the Hessian is

$$\begin{pmatrix} 0 & 4 \\ 4 & 2 \end{pmatrix}$$

which also has determinant -16. So $(4, 0)$ is also a saddle point.
At $(2, 2)$ the Hessian is

$$\begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix}$$

This has determinant $+8$ so the critical point is either a local max or local min. The top-left and bottom-right entries are both positive which means that $(2, 2)$ is a local minimum.

7. (a) $P(RBWR) = 1/45$, $P(RR) = 1/15$
   (b) $P(\text{first three socks different colors}) = 6/6 \times 4/5 \times 2/4 = 2/5$
   $P(\text{first two socks same color}) = 6/6 \times 1/5 = 1/5$.
   (c) The answers from part (b) tell us that $P(X = 4) = 2/5$ and $P(X = 2) = 1/5$. Since the probabilities must add up to 1, this means that $P(X = 3) = 2/5$. Therefore, the expectation of X is

   $$2 \times 1/5 + 3 \times 2/5 + 4 \times 2/5 = 16/5.$$

8. (a) $f(x) = F'(x) = \begin{cases} 2 - 2x & \text{for } 0 \leq x \leq 1; \\ 0 & \text{otherwise.} \end{cases}$ The nonzero part of the graph is a straight line joining the points $(0, 2)$ and $(1, 0)$.
   (b) $P(0.5 \leq X \leq 1) = P(X \leq 1) - P(X \leq 0.5) = 1 - 0.5(2 - 0.5) = 1 - 0.75 = 0.25$
   (c) $EX = \int_0^1 x(2-2x)\,dx = [x^2 - 2x^3/3]_0^1 = 1/3$ and $\operatorname{Var} X = \int_0^1 (x-1/3)^2(2-2x)\,dx$ which is equal to $\int_0^1 (-2x^3 + 10x^2/3 - 14x/9 + 2/9)\,dx = [-2x^4/4 + 10x^3/9 - 14x^2/18 + 2x/9]_0^1 = -1/2 + 10/9 - 7/9 + 2/9 = 1/18$.

9. (a) this is equal to $P(X \leq 2.1) = P(Z \leq (2.1-2.3)/0.1) = P(Z \leq -2) = 1-0.9772 = 0.0228$
   (b) $1 - (0.9772)^16$
   (c) The average time has expectation 2.3 and standard deviation $0.1/\sqrt{16} = 0.025$. Therefore the probability we want is $P(Z \leq (2.225-2.3)/0.025) = P(Z \leq -3) = 1 - 0.9986 = 0.0014$.

10. (a) A 95.44% confidence interval is of the form

    $$\text{sample mean} \pm z \times \text{standard error}$$

    where $P(-z \leq Z \leq z) = 0.9544$. To find z, we rearrange the LHS to get $2P(Z \leq z) - 1 = 0.9544$ or $P(Z \leq z) = 0.9772$. Therefore $z = 2$. The standard error is sample standard deviation divided by $\sqrt{100} = 10$, so is 0.12. Therefore the confidence interval is

    $$10.24 \pm 2 \times 0.12 = [10.00, 10.48].$$

(b) The information in the question tells us that the variance of X is about $1.2^2 = 1.44$, the variance of Y is about $1^2 = 1$, and the variance of $X + Y$ is about 5.2. These numbers are a long way from satisfying $\mathrm{Var}(X+Y) = \mathrm{Var}\,X + \mathrm{Var}\,Y$ which would be the case if X and Y were independent. So it is very unlikely that they are independent (although it is still possible).