110.107 CALCULUS II FINAL EXAMINATION SOLUTIONS 14 December 2001

1. Let $L(t)$ denote the length of a fish at time t and assume that the fish grows according to the von Bertalanffy equation

$$\frac{dL}{dt} = k(40 - L(t)) \text{ with } L(0) = 4$$

If $L(5) = 22$, find the value of the constant k and find the asymptotic length of the fish.

This differential equation is separable: we re-write it and solve:

$$\begin{aligned} \frac{dL}{40 - L} &= kdt \\ -\ln|40 - L| &= kt + C \\ \ln|40 - L| &= -kt + C' \\ |40 - L| &= C''e^{-kt} \end{aligned}$$

Now we apply the condition $L(0) = 4$ into the above result to obtain $40 - 4 = C''$, so

$$|40 - L| = 36e^{-kt}$$

If $0 \leq L(t) \leq 40, L(t) = 40 - 36e^{-kt}$

Finally, the condition $L(5) = 22$ gives:

$$\begin{aligned} L(5) &= 22 \\ 22 &= 40 - 36e^{-5k} \\ 36e^{-5k} &= 18 \\ 2 &= e^{5k} \\ \ln 2 &= 5k \\ k &= \frac{\ln 2}{5} \end{aligned}$$

The asymptotic length is

$$\lim_{t\to\infty} L(t) = \lim_{t\to\infty} 40 - 36e^{-kt} = 40$$

since $k > 0$.

2. Suppose that $N(t)$ denotes the size of a population at time t and that

$$\frac{dN}{dt} = 1.5N\left(1 - \frac{N}{100}\right)$$

Solve this differential equation when $N(0) = 50$ and determine the size of the population in the long run (i.e., calculate $\lim_{t\to\infty} N(t)$).

We re-write this separable equation as follows:

$$\begin{aligned} \frac{dN}{dt} &= 1.5N\left(\frac{100 - N}{100}\right) \\ \frac{100dN}{N(100 - N)} &= 1.5dt \end{aligned}$$

A simple partial fractions decomposition gives

$$\int \frac{100dN}{N(100-N)} = \int \frac{dN}{N} + \int \frac{dN}{100-N} = \ln|N| - \ln|100 - N| = 1.5t + C$$

Simplifying, we get

$$\begin{aligned} \ln\left|\frac{N}{100 - N}\right| &= 1.5t + C \\ \left|\frac{N}{100 - N}\right| &= C'e^{1.5t} \\ \text{If } 0 \le N < 100,& \\ N &= (100 - N)C'e^{1.5t} \\ N &= \frac{100C'e^{1.5t}}{1 + C'e^{1.5t}} \end{aligned}$$

Using the condition $N(0) = 50$ gives $C' = 1$, so

$$N(t) = \frac{100e^{1.5t}}{1 + e^{1.5t}} = \frac{100}{1 + e^{-1.5t}}$$

We see that

$$\lim_{t\to\infty} N(t) = \lim_{t\to\infty} \frac{100}{1 + e^{-1.5t}} = 100$$

since the denominator goes to 1.

3. Let $p = p(t)$ be the fraction of occupied patches in a metapopulation model, and assume that

$$\frac{dp}{dt} = 2p(1 - p) - p \text{ for } t \ge 0$$

Find all equilibria of the above equation that are in the interval [0,1] and determine their stability.

This is an autonomous differential equation with $g(p) = 2p - 2p^2 - p = p(1-2p)$, and so we have equilibria when $p = 0$ and $p = 1/2$. To investigate the stability of these equilibria, we look at $g'(p) = 1-4p$. When $p = 0, g'(0) = 1 > 0$ and so this equilibrium is unstable. When $p = 1/2, g'(1/2) = -1 < 0$ and so this equilibrium is locally stable.

4. Suppose that a drug is administered to a person in a single dose, and assume that the drug does not accumulate in body tissue but is excreted through urine. Let $x_1(t)$ be the concentration in the body after t hours and $x_2(t)$ the concentration in the urine after t hours. If the original dose is 100mg and it takes 30 minutes for the drug to be at one-half of its initial concentration in the body, find a system of differential equations for x_1 and x_2.

   Since the drug flows only one way from the body into the urine, we use the following 2-compartment model:

$$\boxed{x_1} \xrightarrow{ax_1} \boxed{x_2}$$

   This corresponds to the system

$$\begin{aligned} \frac{dx_1}{dt} &= -ax_1 \\ \frac{dx_2}{dt} &= ax_1 \end{aligned}$$

   All we need to do is to figure out the value of a. The easiest way to do this is to actually write down the solution for x_1. This is just $x_1 = ce^{-at}$. Since $x_1(0) = 100$, we have $c = 100$. To find a, we note that after 1/2 hour, x_1 is 50, i.e. $x_1(.5) = 50$. Putting this information into the equation for x_1 gives

$$\begin{aligned} 50 &= 100e^{-a/2} \\ 1/2 &= e^{-a/2} \\ -\ln 2 &= -a/2 \\ a &= 2\ln 2 \end{aligned}$$

   The system is therefore

$$\begin{aligned} \frac{dx_1}{dt} &= -2\ln 2 x_1 \\ \frac{dx_2}{dt} &= 2\ln 2 x_1 \end{aligned}$$

5. A simple two-compartment model for gap dynamics in a forest assumes that gaps are created by disturbances and that they revert back to forest as the trees grow in the gaps. Let $x_1(t)$ denote the area occupied by the gaps, and $x_2(t)$ the area occupied by the adult trees. If the dynamics are given by the system

$$\begin{aligned} \frac{dx_1}{dt} &= -0.3x_1 + 0.2x_2 \\ \frac{dx_2}{dt} &= 0.3x_1 - 0.2x_2 \end{aligned}$$

compute the eigenvalues and eigenvectors for this system, and give the general solution.

We have the matrix equation

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x}$$

where

$$A = \begin{bmatrix} -0.3 & 0.2 \\ 0.3 & -0.2 \end{bmatrix}$$

To find the eigenvalues we calculate

$$\det(A-\lambda I) = \begin{vmatrix} -0.3-\lambda & 0.2 \\ 0.3 & -0.2-\lambda \end{vmatrix} = (-0.3-\lambda)(-0,2-\lambda)-(0.3)(0.2) = \lambda(0.5+\lambda)$$

The eigenvalues are $\lambda_1 = -0.5, \lambda_2 = 0$. Eigenvectors for λ_1 satisfy the system

$$\begin{aligned} -0.3x_1 + 0.2x_2 &= 0.5x_1 \\ 0.3x_1 - 0.2x_2 &= 0.5x_2 \end{aligned}$$

This reduces to $x_1 = -x_2$ and so λ_1 has $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ as an eigenvector. Similarly, eigenvectors for λ_2 satisfy

$$\begin{aligned} -0.3x_1 + 0.2x_2 &= 0 \\ 0.3x_1 - 0.2x_2 &= 0 \end{aligned}$$

This reduces to $0.3x_1 = 0.2x_2$, and so λ_2 has $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ an eigenvector.

The general solution is therefore

$$\mathbf{x}(t) = c_1 e^{-0.5t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

6. Chemotaxis is the chemically directed movement of organisms up a concentration gradient. The slime mold *Dictyostelium discoideum* exhibits this phenomenon; single-celled amoeba of this species move up the concentration gradient of a chemical called cyclic AMP. If the concentration of the cyclic AMP at the point (x,y) in the xy-plane is given by

$$f(x,y) = 6\sqrt{2xy+1}$$

determine in which direction an amoeba at the point (3,4) will move if its movement is directed by chemotaxis.

The direction of movement will be in the direction of the gradient of $f(x)$. We calculate:

$$\nabla f(x,y) = \begin{bmatrix} \frac{\partial f(x,y)}{\partial x} \\ \frac{\partial f(x,y)}{\partial y} \end{bmatrix} = \begin{bmatrix} 6\frac{1}{2}(2xy+1)^{-1/2}2y \\ 6\frac{1}{2}(2xy+1)^{-1/2}2x \end{bmatrix}$$

At $(x,y) = (3,4)$, we have

$$\nabla f(3,4) = \begin{bmatrix} 3(24+1)^{-1/2}8 \\ 3(24+1)^{-1/2}6 \end{bmatrix} = \begin{bmatrix} \frac{24}{5} \\ \frac{18}{5} \end{bmatrix}$$

7. Suppose the mass of a certain animal is normally distributed with a mean of 3750 g and a standard deviation of 350 g. What percentage of the population has a mass between 3225 g and 3925 g?

he lower end of our interval is 3225, which is 1.5 standard deviations below the mean. The right-hand endpoint is 3925, which is 0.5 standard deviations above the mean. The percentage in this interval will be the same as the percentage between the same standard deviation values for the standard normal distribution:

$$\begin{aligned} P &= F(0.5) - F(-1.5) \\ &= F(0.5) - [1 - F(1.5)] \\ &= .6915 - [1 - .9332] \\ &= .6915 - .0668 \\ &= .6247 \end{aligned}$$

8. Consider the exponential distribution

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{for } x > 0 \\ 0 & \text{for } x \leq 0 \end{cases}$$

a) Show that the mean of this distribution is $1/\lambda$.

We have

$$\mu = \int_{-\infty}^{\infty} x f(x)\, dx = \int_{0}^{\infty} \lambda x e^{-\lambda x} dx$$

Integrating by parts with $u = x$ and $dv = \lambda e^{-\lambda x} dx$, we get

$$\int \lambda x e^{-\lambda x} dx = -x e^{-\lambda x} + \int e^{-\lambda x} dx = -x e^{-\lambda x} - \frac{1}{\lambda} e^{-\lambda x} + C$$

The improper integral is thus:

$$\int_0^\infty \lambda x e^{-\lambda x} dx = \lim_{b\to\infty} -b e^{-\lambda b} + 0 - \lim_{b\to\infty} \frac{1}{\lambda} e^{-\lambda b} + \frac{1}{\lambda} = 0 + 0 - 0 + \frac{1}{\lambda} = \frac{1}{\lambda}$$

The first limit requires L'Hopital's Rule to calculate:

$$\lim_{b\to\infty} -b e^{-\lambda b} = \lim_{b\to\infty} \frac{-b}{e^{\lambda b}} = \lim_{b\to\infty} \frac{-1}{\lambda e^{\lambda b}} = 0$$

b) Recall that if $f(x)$ is a continuous distribution, then the *median* is defined as the number m such that

$$\int_{-\infty}^m f(x)\, dx = \int_m^\infty f(x)\, dx = \frac{1}{2}$$

Calculate the median of the exponential distribution above.

Here,

$$\int_{-\infty}^m f(x)\, dx = \int_0^m \lambda e^{-\lambda x} dx = -e^{-\lambda x}\Big|_0^m = -e^{-\lambda m} + 1$$

For this integral to evaluate to 1/2, we require

$$\begin{aligned} -e^{-\lambda m} + 1 &= 1/2 \\ e^{-\lambda m} &= 1/2 \\ -\lambda m &= \ln 1/2 = -\ln 2 \\ \lambda m &= \ln 2 \\ m &= \frac{\ln 2}{\lambda} \end{aligned}$$

9. Suppose that the number of seeds that a plant produces is normally distributed with a mean of 200 and a standard deviation of 25. Find the probability that in a sample of 4 plants, at least one produces more than 250 seeds.

We examine the complementary event that no plant produces more than 250 seeds; i.e., each plant produces no more than 250 seeds. The probability that a plant produces no more than 250 seeds is $F(2)$, since 250 represents two standard deviations above the mean. Since the selection of each plant is independent of the selection of the other plants, the event that all four plants produce no more than 250 seeds is just $F(2)^4$. The complementary event, that at least one plant produces more than 250 seeds, has probability

$$P = 1 - F(2)^4 = 1 - .9772^4 \approx .0882$$