110.107 CALCULUS II
Second MIDTERM EXAM
12 November 2001

Instructions: No calculators or books are allowed. Please do your scratch work on the blank sheets provided. When you are satisfied with your work, write your answers legibly and coherently on the question sheets in the space provided. Justify your answers completely. When in doubt, provide more rather than less explanation.

1. Find the tangent plane to $f(x,y) = \sin x + \cos y$ at $(0,0,1)$.

   The formula is

   $$z - z_0 = \frac{\partial f(x_0,y_0)}{\partial x}(x - x_0) + \frac{\partial f(x_0,y_0)}{\partial y}(y - y_0)$$

   where (x_0, y_0) is $(0,0)$.
   Since $\frac{\partial f(0,0)}{\partial x} = \cos 0 = 1$ and $\frac{\partial f(0,0)}{\partial y} = -\sin 0 = 0$, we havs the equation

   $$z - 1 = 1 \cdot (x - 0) + 0 \cdot (y - 0) \text{ or } z = x + 1$$

2. Find the linear approximation to

   $$\mathbf{f}(x,y) = \begin{bmatrix} e^{xy} \\ \ln xy \end{bmatrix} \text{ at } (1,1)$$

   Let $u = e^{xy}$ and $v = \ln xy$. First we compute the Jacobian:

   $$(D\mathbf{f})(x,y) = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix} = \begin{bmatrix} ye^{xy} & xe^{xy} \\ \frac{1}{x} & \frac{1}{y} \end{bmatrix}$$

   At (1,1), this is the matrix

   $$(D\mathbf{f})(1,1) \begin{bmatrix} e & e \\ 1 & 1 \end{bmatrix}$$

   Now, the linear approximation of $\mathbf{f}(x,y)$ at (1,1) is

   $$L(x,y) = \begin{bmatrix} u(1,1) \\ v(1,1) \end{bmatrix} + (D\mathbf{f})(1,1) \begin{bmatrix} x-1 \\ y-1 \end{bmatrix} = \begin{bmatrix} e \\ 0 \end{bmatrix} + \begin{bmatrix} e & e \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x-1 \\ y-1 \end{bmatrix}$$

3. a) In what direction does $f(x,y) = \sqrt{x^2 - y^2}$ increase most rapidly at (5,3)?

We calculate the gradient of $f(x, y)$:

$$\nabla f(x,y) = \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{bmatrix} = \begin{bmatrix} \frac{1}{2}(x^2-y^2)^{-\frac{1}{2}} \cdot 2x \\ \frac{1}{2}(x^2-y^2)^{-\frac{1}{2}} \cdot -2y \end{bmatrix} = \begin{bmatrix} x(x^2-y^2)^{-\frac{1}{2}} \\ -y(x^2-y^2)^{-\frac{1}{2}} \end{bmatrix}$$

Evaluating at (5,3) we have

$$\nabla f(5,3) = \begin{bmatrix} 5(5^2-3^2)^{-\frac{1}{2}} \\ -3(5^2-3^2)^{-\frac{1}{2}} \end{bmatrix} = \begin{bmatrix} 5(16)^{-\frac{1}{2}} \\ -3(16)^{-\frac{1}{2}} \end{bmatrix} = \begin{bmatrix} 5/4 \\ -3/4 \end{bmatrix}$$

b) What is the value of the directional derivative in this direction at (5,3)?

The directional derivative in the direction of $\begin{bmatrix} 5/4 \\ -3/4 \end{bmatrix}$ is just the dot product of gradient of $f(5,3)$ with the unit vector in the indicated direction. If $\mathbf{v}$ is the vector $\begin{bmatrix} 5/4 \\ -3/4 \end{bmatrix}$, then

$$\mathbf{u} = \frac{\mathbf{v}}{\| \mathbf{v} \|} = \frac{4}{\sqrt{34}} \begin{bmatrix} 5/4 \\ -3/4 \end{bmatrix}$$

is the unit vector in the indicated direction (which happens to be the direction of the gradient).

Now,

$$D_{\mathbf{u}} f(5,3) = \nabla f(5,3) \cdot \mathbf{u} = \begin{bmatrix} 5/4 \\ -3/4 \end{bmatrix} \cdot \frac{4}{\sqrt{34}} \begin{bmatrix} 5/4 \\ -3/4 \end{bmatrix} = \frac{1}{4\sqrt{34}} \cdot 34 = \frac{\sqrt{34}}{4}$$

4. Solve the system of linear equations

$$\begin{aligned} x + y - z &= 0 \\ 2x - y - z &= 0 \\ -x + 2y + z &= 3 \end{aligned}$$

We form the augmented matrix:

$$\left[\begin{array}{rrr|r} 1 & 1 & -1 & 0 \\ 2 & -1 & -1 & 0 \\ -1 & 2 & 1 & 3 \end{array}\right]$$

Replacing the second row with itself minus twice the top row, and the third row with itself plus the top row gives

$$\left[\begin{array}{rrr|r} 1 & 1 & -1 & 0 \\ 0 & -3 & 1 & 0 \\ 0 & 3 & 0 & 3 \end{array}\right]$$

Replacing the second row with itself plus the third row gives

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 3 & 0 & 3 \end{array}\right]$$

Replacing the third row with itself times 1/3 gives

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 1 & 0 & 1 \end{array}\right]$$

Replacing the first with itself minus the third gives

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & -1 \\ 0 & 0 & 1 & 3 \\ 0 & 1 & 0 & 1 \end{array}\right]$$

Replacing the first with itself plus the second gives

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 1 & 0 & 1 \end{array}\right]$$

From this we see the solution $x = 2, z = 3, y = 1$.

5. Compute the eigenvalues λ_1 and λ_2 and corresponding eigenvectors for the matrix

$$A = \begin{bmatrix} 2 & -1 \\ 4 & -3 \end{bmatrix}$$

We calculate $\det(A - \lambda I)$:

$$\det(A - \lambda I) = \begin{vmatrix} 2-\lambda & -1 \\ 4 & -3-\lambda \end{vmatrix} = (2-\lambda)(-3-\lambda) + 4 = \lambda^2 + \lambda - 2 = (\lambda+2)(\lambda-1)$$

The eigenvalues are $\lambda_1 = -2$ and $\lambda_2 = 1$.

To find eigenvectors for $\lambda_1 = -2$ we solve

$$A = \begin{bmatrix} 2 & -1 \\ 4 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -2x \\ -2y \end{bmatrix}$$

which reduces to $4x = y$. One eigenvector is $\begin{bmatrix} 1 \\ 4 \end{bmatrix}$, and the others are constant multiples of this one. Similarly, to find eigenvectors for $\lambda_2 = 1$ we solve

$$A = \begin{bmatrix} 2 & -1 \\ 4 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

which reduces to $x = y$. One eigenvector is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, and the others are constant multiples of this one.

6. Use the matrix A and your answer from the previous problem to compute

$$A^6 \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

We write

$$\begin{bmatrix} 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

This gives us the given vector as a linear combination of two eigencevtors. So,

$$\begin{aligned}
A^6 \begin{bmatrix} 0 \\ 3 \end{bmatrix} &= A^6 \left(\begin{bmatrix} 1 \\ 4 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right) = A^6 \begin{bmatrix} 1 \\ 4 \end{bmatrix} - A^6 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\
&= (-2)^6 \begin{bmatrix} 1 \\ 4 \end{bmatrix} - 1^6 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\
&= 64 \begin{bmatrix} 1 \\ 4 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} 64 \\ 256 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} 63 \\ 255 \end{bmatrix}
\end{aligned}$$