110.107 CALCULUS II
FIRST MIDTERM EXAM
8 October 2001

1. Suppose the amount of phosphorus in a lake at time t, denoted by $P(t)$, follows the equation
$$\frac{dP}{dt} = 3t + 1 \text{ with } P(0) = 0.$$
Find the amount of phosphorous at time $t = 10$.

   In order to find $P(10)$ we must recover P from the differential equation. Separating variables and intgrating gives
$$\int dP = \int 3t + 1\, dt = \frac{3t^2}{2} + t + C$$
or simply
$$P(t) = \frac{3t^2}{2} + t + C$$
Using the initial condition $P(0) = 0$, we have that $C = 0$, so
$$P(t) = \frac{3t^2}{2} + t$$
When $t = 10, P(10) = 300/2 + 10 = 160$.

2. Suppose that a fish population evolves according to the logistic equation, and that a fixed number of fish per unit time are removed. That is,
$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right) - H$$
Assuming that $r = 1$ and $K = 100$, find all possible equilibria and discuss their stability when $H = 24$.

   Substituting in for the constants r, K and H gives the (autonomous) differential equation
$$\frac{dN}{dt} = g(N) = N\left(1 - \frac{N}{100}\right) - 24$$
To find the equilibria, we set $g(N) = 0$ and solve for N:
$$\begin{aligned} g(N) = 0 \quad &\iff \quad N\left(1 - \frac{N}{100}\right) - 24 = 0 \\ &\iff \quad N - N^2/100 - 24 = 0 \\ &\iff \quad \frac{-1}{100}(N^2 - 100N + 2400) = 0 \\ &\iff \quad \frac{-1}{100}(N - 40)(N - 60) = 0 \\ &\iff \quad N = 40 \text{ or } N = 60 \end{aligned}$$

So, there are two equilibria: $N = 40$ and $N = 60$.

To classify these, we investigate $g'(40)$ and $g'(60)$. Since $g'(N) = 1 - N/50$, we see that $g'(40) = 0.2 > 0$ and $g'(60) = -0.2 < 0$). Thus, $N = 40$ is an unstable equilibrium, and $N = 60$ is a locally stable equilibrium.

3. Let the function $f(x)$ be defined as

$$f(x) = \begin{cases} 3e^{-3x} & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases}$$

Show that $f(x)$ is a density function.

We need to show two things: that $f(x) \geq 0$, and that $\int_{-\infty}^{\infty} f(x)dx = 1$.

The first is clear, if x is negative or zero, $f(x) = 0$, and if x is positive, f(x) is positive.

To see that the integral is indeed one, we integrate:

$$\begin{aligned}
\int_{-\infty}^{\infty} f(x)dx &= \int_{-\infty}^{0} f(x)dx + \int_{0}^{\infty} f(x)dx = 0 + \int_{0}^{\infty} f(x)dx \\
&= \int_{0}^{\infty} 3e^{-3x}dx = \lim_{b\to\infty} -e^{-3x}|_0^b \\
&= \lim_{b\to\infty} -e^{-3b} + e^{0} \\
&= 1.
\end{aligned}$$

4. With X distributed according to $f(x)$ in the previous problem, find the mean of X.

Here, we need to compute $\int_{-\infty}^{\infty} xf(x)dx$:

$$\begin{aligned}
\int_{-\infty}^{\infty} xf(x)dx &= \int_{-\infty}^{0} xf(x)dx + \int_{0}^{\infty} xf(x)dx = 0 + \int_{0}^{\infty} xf(x)dx \\
&= \int_{0}^{\infty} x3e^{-3x}dx
\end{aligned}$$

Integrating by parts with $u = x$ (so $du = dx$) and $dv = 3xe^{-3x}dx$ (so $v = -e^{-3x}$) gives

$$\begin{aligned}
\int_{0}^{\infty} x3e^{-3x}dx &= \lim_{b\to\infty} \left[-xe^{-3x} + \int e^{-3x}dx\right]_0^b \\
&= \lim_{b\to\infty} \left[-xe^{-3x} - \frac{1}{3}e^{-3x}\right]_0^b \\
&= \lim_{b\to\infty} \left[-be^{-3b} - \frac{1}{3}e^{-3b} + 0 + \frac{1}{3}e^{-0}\right] \\
&= 1/3
\end{aligned}$$

5. Find the cosine of the angle between the two vectors in the plane

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} \text{ and } \begin{bmatrix} -2 \\ 2 \end{bmatrix}$$

The formula is $\cos\theta = \dfrac{\mathbf{x}\cdot\mathbf{y}}{|\mathbf{x}||\mathbf{y}|}$. Since the numerator is -2 + 4 = 2 and the denominator is $\sqrt{5}\sqrt{8}$ we see that the answer is $\cos\theta = 2/\sqrt{40} = 1/\sqrt{10}$.

6. Suppose a quantitative characteristic is normally distributed with a mean $\mu = 20$ and a standard deviation $\sigma = 3$. Find an interval centered at the mean such that 72 % of the population falls into this interval

Since the interval is centered at the mean, we have that half of the interval, or 36% must fall on either side of the mean. The upper endpoint then has 50% + 36% = 86% of the distribution less than or equal to it. Looking in the table, we see that this means that the upper endpoint of the interval is 1.08 standard deviations above the mean. By symmetry, the lower endpoint must be 1.08 standard deviations below the mean. From the numbers given in the problem, the interval is

$$\mu \pm 1.08\sigma = 20 \pm (1.08)3 = 20 \pm 3.24$$

That is, the interval is $[16.76, 23.24]$.