Math 107: Calculus II, Spring 2006: Final Exam

Thursday May 11, 2005

Give your name and at least one piece of information about your section:

Name:

Section day/time:

Section number (1-6):

TA (Baber/Tasky/Kramer):

1. There are ten questions. Each is worth 20 points.
2. **Do not open your booklet until told to begin.** The exam will be 3 hours long.
3. You may **not** use calculators, books, notes or any other paper. Write all your answers on this booklet. Additional paper is available if required.
4. **Show all your working!** In some questions, points are available for giving reasons for your answers. Unless you are particularly told you only need to give the answer, you should show all your working and explain what you are doing.
5. **READ THE QUESTIONS CAREFULLY!** Do the questions you can do first. Then come back to the ones that look hard.

The following formulas and probabilities may be useful on this exam:

(Binomial distribution) Suppose that the random variable X is binomially distributed with n trials and probability p of success. Then $EX = np$, $\operatorname{Var} X = np(1-p)$ and

$$P(X = k) = \binom{n}{k} p^k (1-p)^{n-k}.$$

(Central Limit Theorem) Suppose that the random variable X has expectation μ and variance σ^2. Then:

- The average of n independent measurements of X is (approximately) normally distributed with mean μ and variance σ^2/n.
- The sum of n independent measurements of X is (approximately) normally distributed with mean $n\mu$ and variance $n\sigma^2$.

(Normal distribution) For a standard normal distribution Z:

- $P(Z \leq 0.1) = 0.5398$
- $P(Z \leq 1) = 0.8413$
- $P(Z \leq 1.28) = 0.9$
- $P(Z \leq 2) = 0.9772$

1. (a) Find the solution to the pure time differential equation

$$\frac{dy}{dt} = 3t^2 + 2t$$

that satisfies the condition $y(0) = 2$.

(b) An autonomous differential equation has a stable equilibrium at $y = 1$ and an unstable equilibrium at $y = 4$. Draw the phase line for this equation and sketch possible solution curves.

(c) Draw the graph of a function $g(y)$ such that the differential equation

$$\frac{dy}{dt} = g(y)$$

would have the behaviour described in part (b).

2. (a) Find the eigenvalues and corresponding eigenvectors of the matrix

$$\begin{pmatrix} 1 & 1 \\ -2 & 4 \end{pmatrix}$$

(b) Use your answers to part (a) to write down the general solution to the system of differential equations

$$\frac{dx}{dt} = x + y, \quad \frac{dy}{dt} = -2x + 4y.$$

(c) The system of equations from part (b) has an equilibrium at $(0, 0)$. Is this equilibrium stable or unstable?

3. (a) Find the inverse of the matrix

$$\begin{pmatrix} 0 & 2 \\ -1 & 1 \end{pmatrix}$$

(b) Solve the following matrix equation

$$\begin{pmatrix} 0 & 2 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}.$$

(c) Find values for the constants a, b such that the matrix equation

$$\begin{pmatrix} 0 & a \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} b \\ 2 \end{pmatrix}$$

has

i. exactly one solution;

ii. infinitely many solutions;

iii. no solutions.

(Note: the three cases are separate – you will find a different pair of values for a, b in each case.)

4. This question concerns the function

$$f(x, y) = x^2 + xy + y.$$

(a) Find the partial derivatives of f.

(b) Find the linear approximation to f at the point $(1, 2)$.

(c) Find the directional derivative of f at $(1, 2)$ in the direction $(-1, 1)$.

(d) Suppose that each of x and y is a function of t. Use the chain rule for multivariable calculus to write down an expression for $\dfrac{df}{dt}$ in terms of x, y, $\dfrac{dx}{dt}$ and $\dfrac{dy}{dt}$.

5. This question concerns the function

$$f(x, y) = 4 - x^2 - y^2.$$

(a) Sketch the c-level curves of f for $c = -5, 0, 3$. (Put all your curves on the same diagram. Make sure you label each curve with the corresponding value of c.)

(b) Calculate the gradient vector ∇f.

(c) Mark on your picture from part (a) the direction of the gradient vector at the points $(0, 1)$, $(2, 0)$ and $(0, -3)$. (Label these three points on your diagram and make sure each appears on the correct level curve.)

(d) In which directions is the directional derivative of f at the point $(-2, 0)$ equal to zero?

6. This question is about critical points of the function

$$f(x, y) = x^2 + kxy + y^2$$

for varying values of the constant k.

(a) Show that for any value of k, the point $(0, 0)$ is a critical point of the function f.

(b) Find the Hessian matrix of f at the critical point $(0, 0)$. (Some of the entries of the matrix will depend on k.)

(c) In each of the following cases, pick a value for k such that the critical point at $(0, 0)$ is of the given type:

i. local minimum;

ii. saddle point.

(d) What type of critical point is $(0, 0)$ in the case $k = 2$? (Hint: the second-derivative test does not work in this case. You need to find another way to determine what type of critical point it is.)

7. The discrete random variable X has the following distribution:

$$P(X=1)=2/5,\ P(X=2)=2/5,\ P(X=3)=0,\ P(X=4)=1/5.$$

(a) Find the expectation and variance of X.

(b) We measure X three times independently. Find the probability that $X=4$ for exactly two of those measurements.

(c) Use the Central Limit Theorem to find the approximate probability that the average of 120 independent measurements of X is larger than 2.2.

8. The continuous random variable X has probability density function given by

$$f(x) = \begin{cases} \frac{2}{x^2} & \text{for } 1 \leq x \leq 2; \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find the cumulative density function $F(x) = P(X \leq x)$.

(b) Find the value x such that $P(X \leq x) = P(X \geq x) = 0.5$.

(c) Calculate the expectation of X.

(d) The continuous random variable Y has probability density function given by

$$f(y) = \begin{cases} 1/4 & \text{for } 0 \leq x \leq 4; \\ 0 & \text{otherwise.} \end{cases}$$

Which of X and Y would you expect to have the larger variance? Explain your answer.

9. A three-sided coin has sides marked $0, 1, 2$ and lands on each side with probability $1/3$. If I flip two such coins, let X be the random variable that measures my total score, and let Y be the random variable given by $Y = 1$ if both coins show the same side and $Y = 2$ if they show different sides.

   (a) Find $P(Y = 1)$.

   (b) Find $P(X = 3)$.

   (c) Find EX and EY. (Make sure you show all your working.)

   (d) Show that X and Y are *not* independent. (It is not enough to explain this only in words. You must do a calculation that shows they are not independent.)

10. An experiment was carried out to estimate the expectation of a random variable X that has a normal distribution. Data from 410 observations had sample mean 20.1 and sample variance 4.1.

   (a) Give an 80% confidence interval for the expectation of X.

   (b) What is the confidence level of the interval [20,20.2]?

   (c) In fact, the expectation of X is 20 and the variance of X is 4. Find $P(20 \leq X \leq 20.2)$.

   (Note to Fall 2006 students: in this class, I do not expect you to be able to do part (b). For part (a), you can use the fact that an 80% confidence interval has the form $\hat{\mu} \pm S.E.$)