

Math 107: Calculus II, Spring 2006: Final Exam Solutions

Please email me if you find a mistake or don't understand any of these solutions.

1. (a) We solve pure time equations just by integrating:

$$y(t) = \int 3t^2 + 2t \, dt = t^3 + 2t + c.$$

We want to have $y(0) = 2$, so we need

$$2 = 0 + 0 + c$$

hence $c = 2$. Therefore the solution is

$$y(t) = t^3 + t^3 + 2.$$

(b) See graphs.

(c) Ditto.

2. (a) To find the eigenvalues of this matrix, we take the determinant of $\begin{pmatrix} 1-k & 1 \\ -2 & 4-k \end{pmatrix}$.

This is $(1-k)(4-k) + 2$ which simplifies to

$$k^2 - 5k + 4 + 2 = (k-2)(k-3).$$

Therefore, the eigenvalues are $k = 2$ and $k = 3$.

To find the eigenvalue corresponding to $k = 2$, we need to solve

$$\begin{pmatrix} -1 & 1 \\ -2 & 2 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This gives $-u + v = 0$ and hence $u = v$. So, for example, $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is an eigenvector with eigenvalue 2.

For $k = 3$, we have

$$\begin{pmatrix} -2 & 1 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

and so $-2u + v = 0$. So, for example, $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ is an eigenvector with eigenvalue 3.

- (b) The general solution is

$$\begin{pmatrix} x \\ y \end{pmatrix} = c_1 e^{2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{3t} \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

- (c) Both eigenvalues are positive, so this is an *unstable* equilibrium.

3. (a) The determinant is $0 \times 1 - 2 \times (-1) = 2$, so the inverse matrix is

$$\frac{1}{2} \begin{pmatrix} 1 & -2 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1/2 & -1 \\ 1/2 & 0 \end{pmatrix}.$$

- (b) Since the matrix on the left-hand side has an inverse, we multiply by that inverse to get the solution:

$$\begin{aligned} \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} 0 & 2 \\ -1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 4 \\ 2 \end{pmatrix} \\ &= \begin{pmatrix} 1/2 & -1 \\ 1/2 & 0 \end{pmatrix} \begin{pmatrix} 4 \\ 2 \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ 2 \end{pmatrix}. \end{aligned}$$

Therefore $x = 0$ and $y = 2$.

- (c) To get exactly one solution, we need to choose a such that the determinant of the matrix on the left-hand side is not zero. Any a not equal to 0 will do, and any value of b will do (e.g. $a = 2$ and $b = 4$ as in part (b)).

To get infinitely many solutions, we need to choose a such that the determinant of the matrix is zero, and $a = 0$ does this. We then need to make sure there is at least one solution which will happen only when $b = 0$. So $a = 0$ and $b = 0$ is the answer.

To get no solutions, we need $a = 0$ as before, and then any b not equal to zero will work.

4. (a) $\frac{\partial f}{\partial x} = 2x + y$ and $\frac{\partial f}{\partial y} = x + 1$.

- (b) The linear approximation to f at the point $(1, 2)$ is given by

$$f(x, y) \simeq f(1, 2) + \frac{\partial f}{\partial x}(1, 2)(x - 1) + \frac{\partial f}{\partial y}(1, 2)(y - 2).$$

We have $f(1, 2) = 5$, $\frac{\partial f}{\partial x}(1, 2) = 4$ and $\frac{\partial f}{\partial y}(1, 2) = 2$, so we get

$$f(x, y) \simeq 5 + 4(x - 1) + 2(y - 2).$$

- (c) (Note: I used slightly different notation in the class last semester. The direction $(-1, 1)$ should be written as a vector $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ so as not to be confused with the *point* $(-1, 1)$.)

The directional derivative in direction \mathbf{u} is given by

$$\frac{\nabla f(1, 2) \cdot \mathbf{u}}{|\mathbf{u}|}$$

We have $\nabla f(1, 2) = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$ and so the directional derivative we want is

$$\frac{\begin{pmatrix} 4 \\ 4 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 1 \end{pmatrix}}{\sqrt{2}} = \frac{-2}{\sqrt{2}} = -\sqrt{2}.$$

(d) $\frac{df}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt} = (2x + y) \frac{dx}{dt} + (x + 1) \frac{dy}{dt}.$

5. (a) For $c = -5$, the level curve is given by $x^2 + y^2 = 9$ which is a circle of radius 3 about the origin. For $c = 0$, it is a circle of radius 2 and for $c = 3$ is a circle of radius 1. See the graph pages for pictures.

(b) $\nabla f = \begin{pmatrix} -2x \\ -2y \end{pmatrix}.$

(c) See graphs.

- (d) The directional derivative is zero in the direction parallel to the level curves (or perpendicular to the gradient vectors). At the point $(-2, 0)$, these are the vertical directions (both up and down).

6. (a) The partial derivatives are

$$\frac{\partial f}{\partial x} = 2x + ky, \quad \frac{\partial f}{\partial y} = kx + 2y.$$

Substituting in $x = 0$ and $y = 0$, we get 0 in both cases, so $(0, 0)$ is a critical point.

- (b)

$$Hf(0, 0) = \begin{pmatrix} 2 & k \\ k & 2 \end{pmatrix}.$$

- (c) Since the top-left and bottom-right entries are positive, we will get a local minimum when the determinant of Hf is positive and a saddle point when the determinant is negative. The determinant is $4 - k^2$, so, for example, $k = 0$ gives a local minimum and $k = 3$ gives a saddle point.

- (d) When $k = 2$, we can rewrite the function as

$$f(x, y) = x^2 + 2xy + y^2 = (x + y)^2.$$

This is the square of something, so it is always greater than or equal to 0. Therefore $(0, 0)$ is a local (and in fact global) minimum.

7. (a) The expectation is

$$EX = 1 \times 2/5 + 2 \times 2/5 + 4 \times 1/5 = 2.$$

The variance is

$$\text{Var } X = (1 - 2)^2 \times 2/5 + (2 - 2)^2 \times 2/5 + (4 - 2)^2 \times 1/5 = 6/5 = 1.2.$$

- (b) The number of times you get 4 from three measurements is binomially distributed with $n = 3$ and $p = P(X = 4) = 1/5$. Therefore the probability of getting 4 exactly twice is

$$\binom{3}{2} \left(\frac{1}{5}\right)^2 \left(\frac{4}{5}\right) = \frac{12}{125}.$$

- (c) By the CLT, the average \bar{X} is (approximately) normally distributed with mean 2 (the same as X) and variance $\sigma^2/n = 1.2/120 = 0.01$. The standard deviation of \bar{X} is therefore $\sqrt{0.01} = 0.1$. So:

$$P(\bar{X} > 2.2) = P\left(Z > \frac{2.2 - 2}{0.1}\right) = P(Z > 2) = 1 - P(Z < 2) = 1 - 0.9772 = 0.0228.$$

8. (a) The range of possible values for X is $1 \leq X \leq 2$. Therefore:

$$F(x) = \int_1^x \frac{2}{x^2} dx = [-2/x]_1^x = \frac{-2}{x} + 2.$$

Note that the lower limit of integration is 1 because that is the lower end of the range of possible values of X .

- (b) We need $-2/x + 2 = 0.5$, so $-2/x = -1.5$, so $x = 2/1.5 = 4/3$.
 (c) The expectation of X is

$$EX = \int_1^2 x \frac{2}{x^2} dx = \int_1^2 \frac{2}{x} dx = [2 \ln |x|]_1^2 = 2 \ln 2 - 2 \ln 1 = 2 \ln 2.$$

- (d) The range of possible values for Y is $0 \leq Y \leq 4$, i.e. it is spread over a much wider range than X . Therefore, you would expect Y to have the larger variance.
 9. (a) Each outcome of the experiment (i.e. 00, 01, 02, 10, ...) has probability $1/9$. Therefore

$$\begin{aligned} P(Y = 1) &= P(\text{coins show same side}) \\ &= P(00) + P(11) + P(22) \\ &= 1/9 + 1/9 + 1/9 \\ &= 1/3. \end{aligned}$$

- (b) $P(X = 3) = P(12) + P(21) = 2/9$.

- (c) To do this we have to find the distribution of X and Y , i.e. all the probabilities that they take certain values. For X :

$$\begin{aligned} P(X = 0) &= 1/9, P(X = 1) = 2/9, P(X = 2) = 3/9, \\ P(X = 3) &= 2/9, P(X = 4) = 1/9. \end{aligned}$$

Therefore

$$EX = 0 \times 1/9 + 1 \times 2/9 + 2 \times 3/9 + 3 \times 2/9 + 4 \times 1/9 = 2.$$

For Y , we have $P(Y = 1) = 1/3$ and $P(Y = 2) = 2/3$. Therefore $EY = 1 \times 1/3 + 2 \times 2/3 = 5/3$.

- (d) $P(Y = 1 \text{ and } X = 3) = 0 \neq P(Y = 1)P(X = 3)$. This means that X and Y are not independent.
10. (a) Note that we are given the sample variance (not the sample standard deviation), so the standard error is given by

$$S.E. = \frac{\sqrt{4.1}}{\sqrt{n}} = \sqrt{\frac{4.1}{410}} = \sqrt{0.01} = 0.1.$$

The hint should have said $\hat{\mu} \pm 1.28 \times S.E.$ (not $\hat{\mu} \pm S.E.$). With this correction, the confidence interval is

$$20.1 \pm 1.28 \times 0.1 = 20.1 \pm 0.128.$$

Without the correction, you would get

$$20.1 \pm 0.1.$$

- (b) You do not need to be able to do this - the answer is that it is a confidence level of 68.26%.
- (c)

$$\begin{aligned} P(20 \leq X \leq 20.2) &= P\left(0 \leq Z \leq \frac{20.2 - 20}{\sqrt{4}}\right) \\ &= P(0 \leq Z \leq 0.1) \\ &= P(Z \leq 0.1) - P(Z \leq 0) \\ &= 0.5398 - 0.5 \\ &= 0.0398. \end{aligned}$$