Math 107, Spring 2006: Midterm I Practice Exam Solutions

1. *(a) Find the general solution to the differential equation*

$$\frac{dy}{dt} = 3t^2 + e^t.$$

(b) Find the specific solution to this equation that has the initial condition

$$y(0) = 3.$$

(a) This is a pure time differential equation, so the solution is just the general antiderivative of the function of t on the right-hand side. We find this by integrating:

$$y(t) = t^3 + e^t + c,$$

not forgetting the constant of integration of course.

(b) To find the specific solution, we plug in $t = 0$ and $y = 3$ to get

$$3 = 0 + 1 + c$$

so $c = 2$. Therefore, the specific solution with this initial condition is:

$$y(t) = t^3 + e^t + 2.$$

2. *Consider the autonomous differential equation*

$$\frac{dy}{dt} = g(y)$$

where $g(y) = (1 - y)y$.

(a) Find the equilibrium solutions of this equation.

(b) Use the first derivative test to classify the stability of these equilibria.

(c) Sketch the solutions to this differential equation.

(a) The equilibrium solutions occur where $\frac{dy}{dt} = 0$, that is $(1 - y)y = 0$. So the equilibrium solutions are $y = 0$ and $y = 1$ (the constant functions).

(b) To use the first derivative test, we have to differentiate the function $g(y)$. Since $g(y) = y - y^2$ in this problem, we get

$$g'(y) = 1 - 2y.$$

Now we plug in the equilibria:

$$g'(0) = 1 - 2 \times 0 = 1$$

which is positive. This means that the equilibrium $y = 0$ is **unstable**. Next:

$$g'(1) = 1 - 2 \times 1 = -1$$

which is negative. This means the equilibrium $y = 1$ is **stable**.

(c) I don't have the means to draw the solutions on the computer. You'll have to come and see me so I can draw this graph for you. Note though that this is asking for the graph of y against t. There will be lots of curves on the graph because there are lots of different solutions to a differential equation.

3. (a) *Find the determinant of the matrix* $\begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}$.

   (b) *Find the inverse of the matrix from part (a).*

   (c) *Use your answer to (b) to solve the system of linear equations*

$$\begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 8 \\ 16 \end{pmatrix}.$$

(a) The determinant is given by the formula "$ad - bc$". In this case, that is $1 \times 1 - 3 \times 3 = -8$.

(b) The formula for the inverse matrix is:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

so in this case we get

$$\frac{1}{-8} \begin{pmatrix} 1 & -3 \\ -3 & 1 \end{pmatrix}.$$

(c) When the 2x2 matrix is invertible, we solve just by multiplying the other side by the inverse. Therefore:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{-8} \begin{pmatrix} 1 & -3 \\ -3 & 1 \end{pmatrix} \begin{pmatrix} 8 \\ 16 \end{pmatrix} = \frac{1}{-8} \begin{pmatrix} -40 \\ -8 \end{pmatrix} = \begin{pmatrix} 5 \\ 1 \end{pmatrix}.$$

4. (a) *Show that* $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ *and* $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ *are eigenvectors of the matrix* $\begin{pmatrix} 2 & 0 \\ 1 & 3 \end{pmatrix}$, *and find the corresponding eigenvalues.*

   (b) *Find the general solution to the following system of differential equations (write your answer in the form* $x(t) = \dots, y(t) = \dots$*):*

$$\frac{dx}{dt} = 2x; \quad \frac{dy}{dt} = x + 3y.$$

(a) To show that something is an eigenvector, we just have to multiply it by the matrix and notice that we get a multiple of what we started with:

$$\begin{pmatrix} 2 & 0 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \times 0 + 0 \times 1 \\ 1 \times 0 + 3 \times 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \end{pmatrix} = 3 \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Therefore, $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is an eigenvector of $\begin{pmatrix} 2 & 0 \\ 1 & 3 \end{pmatrix}$ with eigenvalue 3.

Similarly,

$$\begin{pmatrix} 2 & 0 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \end{pmatrix} = 2 \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

so $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is an eigenvector with eigenvalue 2.

(b) We can use the eigenvalues and eigenvectors to write down the general solution to the system – no work is required. The solution is:

$$\begin{pmatrix} x \\ y \end{pmatrix} = c_1 e^{3t} \begin{pmatrix} 0 \\ 1 \end{pmatrix} + c_2 e^{2t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} c_2 e^{2t} \\ 3c_1 e^{3t} - c_2 e^{2t} \end{pmatrix}.$$

Therefore:

$$x(t) = c_2 e^{2t}; \quad y(t) = 3c_1 e^{3t} - c_2 e^{2t}.$$

5. *A colony of penguins has $P(t)$ members at time t (measured in years). The colony's size varies as a result of reproduction and penguins dying. If there are fewer than 10 members, they will all die off. If there are more than 10 members, the size of the colony will increase, up to a maximum of 100. At either 10 or 100 members, the colony is in equilibrium.*

(a) *The colony's size satisfies a differential equation*

$$\frac{dP}{dt} = g(P)$$

for some function $g(P)$. Draw the graph of a possible function $g(P)$ based on the information given about how the size of the colony varies.

(b) *Are the equilibria stable or unstable?*

(a) Again, I don't have the means to draw the graph right now. The key features are:

- It is a graph of $g(P)$ against P.
- For $0 < P < 10$, $g(P) < 0$.
- For $P = 10$ or $P = 100$, $g(P) = 0$.
- For $10 < P < 100$, $g(P) > 0$.

Since the question did not explicitly say what happened if there are more than 100 penguins, your graph does not have to display what happens for $P > 100$ (and $P < 0$ is unnecessary because we cannot have negative penguins).

(b) The equilibrium at $P = 10$ is unstable, and the equilibrium at $P = 100$ is stable. This can be read directly from the information given in the question. Another way to see this is that your graph should have negative gradient at $P = 100$ and positive gradient at $P = 10$. (The stability result then follows from the first derivative test.)