Math 107: Calculus II, Spring 2008: Practice Final III Solutions

1. (a) We solve pure time equations just by integrating:

$$y(t) = \int 3t^2 + 2t\,dt = t^3 + t^2 + c.$$

We want to have $y(0) = 2$, so we need

$$2 = 0 + 0 + c$$

hence $c = 2$. Therefore the solution is

$$y(t) = t^3 + t^2 + 2.$$

(b) See separate file for graphs.

(c) Ditto.

2. (a) To find the eigenvalues of this matrix, we take the determinant of $\begin{pmatrix} 1-k & 1 \\ -2 & 4-k \end{pmatrix}$. This is $(1-k)(4-k)+2$ which simplifies to

$$k^2 - 5k + 4 + 2 = (k-2)(k-3).$$

Therefore, the eigenvalues are $k = 2$ and $k = 3$.
To find the eigenvalue corresponding to $k = 2$, we need to solve

$$\begin{pmatrix} -1 & 1 \\ -2 & 2 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This gives $-u + v = 0$ and hence $u = v$. So, for example, $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is an eigenvector with eigenvalue 2.
For $k = 3$, we have

$$\begin{pmatrix} -2 & 1 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

and so $-2u + v = 0$. So, for example, $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ is an eigenvalue with eigenvalue 3.

(b) The general solution is

$$\begin{pmatrix} x \\ y \end{pmatrix} = c_1 e^{2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{3t} \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

(c) Both eigenvalues are positive, so this is an *unstable* equilibrium.

3. (a) The determinant is $0 \times 1 - 2 \times (-1) = 2$, so the inverse matrix is

$$\frac{1}{2}\begin{pmatrix} 1 & -2 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1/2 & -1 \\ 1/2 & 0 \end{pmatrix}.$$

(b) Since the matrix on the left-hand side has an inverse, we multiply by that inverse to get the solution:

$$\begin{aligned} \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} 0 & 2 \\ -1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 4 \\ 2 \end{pmatrix} \\ &= \begin{pmatrix} 1/2 & -1 \\ 1/2 & 0 \end{pmatrix} \begin{pmatrix} 4 \\ 2 \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ 2 \end{pmatrix}. \end{aligned}$$

Therefore $x = 0$ and $y = 2$.

(c) To get exactly one solution, we need to choose a such that the determinant of the matrix on the left-hand side is not zero. Any a not equal to 0 will do, and any value of b will do (e.g. $a = 2$ and $b = 4$ as in part (b)).

To get infinitely many solutions, we need to choose a such that the determinant of the matrix *is* zero, and $a = 0$ does this. We then need to make sure there is at least one solution which will happen only when $b = 0$. So $a = 0$ and $b = 0$ is the answer.

To get no solutions, we need $a = 0$ as before, and then any b not equal to zero will work.

4. (a) $\dfrac{\partial f}{\partial x} = 2x + y$ and $\dfrac{\partial f}{\partial y} = x + 1$.

(b) The linear approximation to f at the point $(1, 2)$ is given by

$$f(x, y) \simeq f(1, 2) + \frac{\partial f}{\partial x}(1, 2)(x - 1) + \frac{\partial f}{\partial y}(1, 2)(y - 2).$$

We have $f(1, 2) = 5$, $\frac{\partial f}{\partial x}(1, 2) = 4$ and $\frac{\partial f}{\partial y}(1, 2) = 2$, so we get

$$f(x, y) \simeq 5 + 4(x - 1) + 2(y - 2).$$

(c) (Note: I used slightly different notation in the class last semester. The direction $(-1, 1)$ should be written as a vector $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ so as not to be confused with the *point* $(-1, 1)$.)

The directional derivative in direction $\mathbf{u}$ is given by

$$\frac{\nabla f(1, 2) \cdot \mathbf{u}}{|\mathbf{u}|}$$

We have $\nabla f(1,2) = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$ and so the directional derivative we want is

$$\frac{\begin{pmatrix} 4 \\ 4 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 1 \end{pmatrix}}{\sqrt{2}} = \frac{-2}{\sqrt{2}} = -\sqrt{2}.$$

5. (a) For $c = -5$, the level curve is given by $x^2 + y^2 = 9$ which is a circle of radius 3 about the origin. For $c = 0$, it is a circle of radius 2 and for $c = 3$ is is a circle of radius 1. See the graph pages at the end of these solutions for pictures.

   (b) $\nabla f = \begin{pmatrix} -2x \\ -2y \end{pmatrix}$.

   (c) See separate file for graphs.

   (d) The directional derivative is zero in the direction parallel to the level curves (or perpendicular to the gradient vectors). At the point $(-2, 0)$, these are the vertical directions (both up and down).

6. (a) The partial derivatives are

$$\frac{\partial f}{\partial x} = 2x + ky, \quad \frac{\partial f}{\partial y} = kx + 2y.$$

   Substituting in $x = 0$ and $y = 0$, we get 0 in both cases, so $(0, 0)$ is a critical point.

   (b)

$$Hf(0,0) = \begin{pmatrix} 2 & k \\ k & 2 \end{pmatrix}.$$

   (c) Since the top-left and bottom-right entries are positive, we will get a local minimum when the determinant of Hf is positive and a saddle point when the determinant is negative. The determinant is $4 - k^2$, so, for example, $k = 0$ gives a local minimum and $k = 3$ gives a saddle point.

   (d) When $k = 2$, we can rewrite the function as

$$f(x, y) = x^2 + 2xy + y^2 = (x + y)^2.$$

   This is the square of something, so it is always greater than or equal to 0. Therefore $(0, 0)$ is a local (and in fact global) minimum.

7. (a) The expectation is

$$EX = 1 \times 2/5 + 2 \times 2/5 + 4 \times 1/5 = 2.$$

   The variance is

$$\operatorname{Var} X = (1-2)^2 \times 2/5 + (2-2)^2 \times 2/5 + (4-2)^2 \times 1/5 = 6/5 = 1.2.$$

(b) The number of times you get 4 from three measurements is binomially distributed with $n = 3$ and $p = P(X = 4) = 1/5$. Therefore the probability of getting 4 exactly twice is

$$\binom{3}{2}\left(\frac{4}{5}\right)^2\left(\frac{4}{5}\right) = \frac{12}{25}.$$

(c) By the CLT, the average $\bar{X}$ is (approximately) normally distributed with mean 2 (the same as X) and variance $\sigma^2/n = 1.2/120 = 0.01$. The standard deviation of $\bar{X}$ is therefore $\sqrt{0.01} = 0.1$. So:

$$P(\bar{X} > 2.2) = P(Z > \frac{2.2-2}{0.1}) = P(Z > 2) = 1 - P(Z < 2) = 1 - 0.9772 = 0.0228.$$

8. (a) The range of possible values for X is $1 \le X \le 2$. Therefore:

$$F(x) = \int_1^x \frac{2}{x^2}\,dx = [-2/x]_1^x = \frac{-2}{x} + 2.$$

Note that the lower limit of integration is 1 because that is the lower end of the range of possible values of X.

(b) We need $-2/x + 2 = 0.5$, so $-2/x = -1.5$, so $x = 2/1.5 = 4/3$.

(c) The expectation of X is

$$EX = \int_1^2 x\frac{2}{x^2}\,dx = \int_1^2 \frac{2}{x}\,dx = [2\ln|x|]_1^2 = 2\ln 2 - 2\ln 1 = 2\ln 2.$$

(d) The range of possible values for Y is $0 \le Y \le 4$, i.e. it is spread over a much wider range than X. Therefore, you would expect Y to have the larger variance.

9. (a) Each outcome of the experiment (i.e. $00, 01, 02, 10, \ldots$) has probability $1/9$. Therefore

$$\begin{aligned} P(Y = 1) &= P(\text{coins show same side}) \\ &= P(00) + P(11) + P(22) \\ &= 1/9 + 1/9 + 1/9 \\ &= 1/3. \end{aligned}$$

(b) $P(X = 3) = P(12) + P(21) = 2/9$.

(c) To do this we have to find the distribution of X and Y, i.e. all the probabilities that they take certain values. For X:

$$P(X = 0) = 1/9, P(X = 1) = 2/9, P(X = 2) = 3/9,$$
$$P(X = 3) = 2/9, P(X = 4) = 1/9.$$

Therefore

$$EX = 0 \times 1/9 + 1 \times 2/9 + 2 \times 3/9 + 3 \times 2/9 + 4 \times 1/9 = 2.$$

For Y, we have $P(Y = 1) = 1/3$ and $P(Y = 2) = 2/3$. Therefore $EY = 1 \times 1/3 + 2 \times 2/3 = 5/3$.

(d) $P(Y = 1 \text{ and } X = 3) = 0 \neq P(Y = 1)P(X = 3)$. This means that X and Y are not independent.

(This is a particularly nasty question because in this case (if my calculations are correct) we have $E(XY) = (EX)(EY)$, even though X and Y are *not* independent. I didn't stress this in class very much so you won't get a problem like this, but the point is that *if* X and Y are independent *then* $E(XY) = (EX)(EY)$, but the other way round is not necessarily true: i.e. $E(XY) = (EX)(EY)$ does not necessarily mean that X and Y are independent.)

10. (a) The best estimate for the probability of heads is $\hat{p} = 13/40$. The standard error with this estimate is

$$\text{S.E.} = \sqrt{\frac{\hat{p}(1-\hat{p})}{n-1}} = \sqrt{\frac{13/40 \times 27/40}{39}} = \sqrt{\frac{9}{40 \times 40}} = \frac{3}{40} = 0.075$$

Therefore, a 95% confidence interval is given by

$$\frac{13}{40} \pm 1.96 \times 0.075 = 0.325 \pm 0.147.$$

So the confidence interval is $[0.178, 0.472]$.

(b) The normal approximation to the binomial distribution says that the number of heads is approximately a normal distribution with $\mu = np = 40 \times 0.4 = 16$ and $\sigma = \sqrt{np(1-p)} = \sqrt{40 \times 0.4 \times 0.6} = \sqrt{9.6}$. With the histogram correction, we want to find $P(12.5 \leq X \leq 13.5)$ which is therefore equal to

$$\begin{aligned} &P((12.5 - 16)/\sqrt{9.6} \leq Z \leq (13.5 - 16)/\sqrt{9.6}) \\ &= P(Z \leq -2.5/\sqrt{9.6}) - P(Z \leq -3.5/\sqrt{9.6}). \end{aligned}$$